Krylov Subspace Methods

1a. Consider the matrix defined by
\[
a_{ij} = \begin{cases} 
4 & \text{if } i = j \\
-1 & \text{if } i = j + 1 \text{ or } i + 1 = j \\
0 & \text{otherwise.}
\end{cases}
\]
Let \( n = 1000 \) and write a Matlab program to find \( Ax \) for any vector \( x \in \mathbb{R}^n \). Alternatively, read about sparse matrices in Matlab and a figure how to represent \( A \) using a sparse matrix.

1b. Let \( b \) be the vector in \( \mathbb{R}^n \) defined by \( b_i = 1/i^2 \) and define
\[
K_m = \text{span}\{b, Ab, A^2b, \ldots, A^{m-1}b\}.
\]
Let \( x_m \in K_m \) be a vector which minimizes \( \|Ax - b\| \) among all possible vectors in \( K_m \). Compute \( \|Ax_m - b\| \) for \( m = 1, 2, \ldots, 10 \).

1c. Let \( x^* \) be the best approximation to \( Ax = b \) found in step 1b. Define the residual \( r = b - Ax^* \) and let \( y^* \) be the point in
\[
\text{span}\{r, Ar, A^2r, \ldots, A^9r\}
\]
which minimizes \( \|Ay - r\| \). Find the norm of the error \( \|A(x^* + y^*) - b\| \) for the iteratively improved solution \( x^* + y^* \) to \( Ax = b \).

1d. Let \( n = 100 \) and repeat step 1b for the matrix \( A \) and vector \( b \) given by \( A=\text{randn}(100) \) and \( b=\text{rand}(100,1) \). Does the Krylov subspace method of minimum residuals work to solve \( Ax = b \) in this case? Use the Matlab command \( \text{plot}(	ext{eig}(A),'+') \) to plot the eigenvalues of \( A \).

1e. Using the matrix \( A \) from part d define \( B = A + 20I \). Plot the eigenvalues of \( B \). How are they related to the eigenvalues of \( A \)? Does the Krylov subspace method work to solve \( Bz = b \)? What is the relation between \( z \) and the solution \( x \) of \( Ax = b \)? Can you find \( x \) from \( z \) without inverting \( A \)?

1f. [Extra Credit and for CS/Math 666] Repeat the steps 1a–c for the 5000 × 5000 matrix defined by
\[
a_{ij} = \begin{cases} 
8 & \text{if } i = j \\
-1 & \text{if } i = j + 1 \text{ or } i + 1 = j \\
-1/2 & i = 2j + 1 \text{ or } 2i + 1 = j \\
-1/4 & i = 3j + 2 \text{ or } 3i + 2 = j \\
0 & \text{otherwise.}
\end{cases}
\]
Print using \texttt{format long} the 1-st component and the 1000-th component of the vectors \( x^* \) and \( y^* \) for reference.