From Swokowski, Algebra and Trigonometry with Analytic Geometry

4.3 Properties of Division

In the following discussion, symbols such as $f(x)$ and $g(x)$ will be used to denote polynomials in x. If a polynomial $g(x)$ is a factor of a polynomial $f(x)$, then $f(x)$ is said to be **divisible** by $g(x)$. For example, the polynomial $x^4 - 16$ is divisible by $x^2 - 4$, by $x^2 + 4$, by $x + 2$, and by $x - 2$; but $x^2 + 3x + 1$ is not a factor of $x^4 - 16$. However, by *long division*, we can write

$$
\begin{array}{r|l}
 & x^2 - 3x + 8 \\
\hline
x^2 + 3x + 1 & x^4 \qquad\qquad\qquad\qquad - 16 \\
 & x^4 + 3x^3 + x^2 \\
\cline{2-2}
 & -3x^3 - x^2 \\
 & -3x^3 - 9x^2 - 3x \\
\cline{2-2}
 & 8x^2 + 3x - 16 \\
 & 8x^2 + 24x + 8 \\
\cline{2-2}
 & -21x - 24
\end{array}
$$

The polynomial $x^2 - 3x + 8$ is called the **quotient** and $-21x - 24$ is the **remainder.**

Note that the long division process ends when we arrive at a polynomial (the remainder) that either is 0 or has smaller degree than the divisor. The result of this division is often written

$$\frac{x^4 - 16}{x^2 + 3x + 1} = (x^2 - 3x + 8) + \left(\frac{-21x - 24}{x^2 + 3x + 1}\right).$$

Multiplying both sides of the equation by $x^2 + 3x + 1$, we obtain

$$x^4 - 16 = (x^2 + 3x + 1)(x^2 - 3x + 8) + (-21x - 24).$$

This example illustrates the following theorem, which we state without proof.

DIVISION ALGORITHM FOR POLYNOMIALS

If $f(x)$ and $g(x)$ are polynomials and if $g(x) \neq 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x)$$

where either $r(x) = 0$, or the degree of $r(x)$ is less than the degree of $g(x)$. The polynomial $q(x)$ is called the **quotient**, and $r(x)$ is the **remainder** in the division of $f(x)$ by $g(x)$.

An interesting special case occurs if $f(x)$ is divided by a polynomial of the form $x - c$ where c is a real number. If $x - c$ is a factor of $f(x)$, then

$$f(x) = (x - c)q(x)$$

for some polynomial $q(x)$; that is, the remainder $r(x)$ is 0. If $x - c$ is not a factor of $f(x)$, then the degree of the remainder $r(x)$ is less than the degree of $x - c$, and hence $r(x)$ must have degree 0. This, in turn, means that the remainder is a nonzero number. Consequently, in all cases we have

$$f(x) = (x - c)q(x) + d$$

where the remainder d is some real number (possibly $d = 0$). If c is substituted for x in the equation $f(x) = (x - c)q(x) + d$, we obtain

$$f(c) = (c - c)q(c) + d,$$

which reduces to $f(c) = d$. This proves the following theorem.

REMAINDER THEOREM

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

EXAMPLE 1 If $f(x) = x^3 - 3x^2 + x + 5$, use the Remainder Theorem to find $f(2)$.

SOLUTION According to the Remainder Theorem, $f(2)$ is the remainder when $f(x)$ is divided by $x - 2$. By long division,

$$\begin{array}{r|l} & x^2 - x - 1 \\ \hline x - 2 & x^3 - 3x^2 + x + 5 \\ & x^3 - 2x^2 \\ \hline & - x^2 + x \\ & - x^2 + 2x \\ \hline & - x + 5 \\ & - x + 2 \\ \hline & 3 \end{array}$$

Hence, $f(2) = 3$. We may check this fact by direct substitution. Thus, $f(2) = 2^3 - 3(2)^2 + 2 + 5 = 3$. ■

FACTOR THEOREM

> A polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$.

PROOF By the Remainder Theorem, $f(x) = (x - c)q(x) + f(c)$ for some quotient $q(x)$. If $f(c) = 0$, then $f(x) = (x - c)q(x)$; that is, $x - c$ is a factor of $f(x)$. Conversely, if $x - c$ is a factor, then the remainder upon division of $f(x)$ by $x - c$ must be 0, and hence, by the Remainder Theorem, $f(c) = 0$. □

The Factor Theorem is useful for finding factors of polynomials, as illustrated in the next example.

EXAMPLE 2 Show that $x - 2$ is a factor of the polynomial

$$f(x) = x^3 - 4x^2 + 3x + 2.$$

SOLUTION Since $f(2) = 8 - 16 + 6 + 2 = 0$, it follows from the Factor Theorem that $x - 2$ is a factor of $f(x)$. Of course, another method of solution would be to divide $f(x)$ by $x - 2$ and show that the remainder is 0. The quotient in the division would be another factor of $f(x)$. ■

EXAMPLE 3 Find a polynomial $f(x)$ of degree 3 that has zeros 2, -1, and 3.

SOLUTION By the Factor Theorem, $f(x)$ has factors $x - 2$, $x + 1$, and $x - 3$. We may then write

$$f(x) = a(x - 2)(x + 1)(x - 3)$$

where any nonzero value may be assigned to a. If we let $a = 1$ and multiply, we obtain

$$f(x) = x^3 - 4x^2 + x + 6.$$

To apply the Remainder Theorem it is necessary to divide a given polynomial by $x - c$. A method called **synthetic division** may be used to simplify this work. The following rules state how to proceed. The method can be justified by a careful (and lengthy) comparison with the method of long division.

Synthetic Division of $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ by $x - c$

1. Begin with the following display, supplying zeros for any missing coefficients in the given polynomial.

$$\begin{array}{r|llllll} c & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ \hline & a_n & & & & & \end{array}$$

2. Multiply a_n by c and place the product ca_n underneath a_{n-1} as indicated by the arrow in the following display. (This arrow, and others, is used only to help clarify these rules, and will not appear in *specific* synthetic divisions.) Next find the sum $b_1 = a_{n-1} + ca_n$ and place it below the line as shown.

$$\begin{array}{r|lllllll} c & a_n & a_{n-1} & a_{n-2} & \cdots & & a_1 & a_0 \\ & & ca_n & cb_1 & cb_2 & \cdots & cb_{n-2} & cb_{n-1} \\ \hline & a_n & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} & r \end{array}$$

3. Multiply b_1 by c and place the product cb_1 underneath a_{n-2} as indicated by another arrow. Next find the sum $b_2 = a_{n-2} + cb_1$ and place it below the line as shown.

4. Continue this process, as indicated by the arrows, until the final sum $r = a_0 + cb_{n-1}$ is obtained. The numbers

$$a_n, \quad b_1, \quad b_2, \quad \ldots, \quad b_{n-2}, \quad b_{n-1}$$

are the coefficients of the quotient $q(x)$; that is,

$$q(x) = a_n x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2}x + b_{n-1},$$

and r is the remainder.

The following examples illustrate synthetic division for some special cases.

EXAMPLE 4 Use synthetic division to find the quotient and remainder if $2x^4 + 5x^3 - 2x - 8$ is divided by $x + 3$.

SOLUTION Since the divisor is $x + 3$, the c in the expression $x - c$ is -3. Hence, the synthetic division takes this form:

−3	2	5	0	−2	−8
		−6	3	−9	33
	2	−1	3	−11	25

The first four numbers in the third row are the coefficients of the quotient $q(x)$ and the last number is the remainder r. Thus,

$$q(x) = 2x^3 - x^2 + 3x - 11 \quad \text{and} \quad r = 25.$$

Synthetic division can be used to find values of polynomial functions, as illustrated in the next example.

EXAMPLE 5 If $f(x) = 3x^5 - 38x^3 + 5x^2 - 1$, use synthetic division to find $f(4)$.

SOLUTION By the Remainder Theorem, $f(4)$ is the remainder when $f(x)$ is divided by $x - 4$. Dividing synthetically, we obtain

4	3	0	−38	5	0	−1
		12	48	40	180	720
	3	12	10	45	180	719

Consequently, $f(4) = 719$.

Synthetic division may be employed to help find zeros of polynomials. By the method illustrated in the preceding example, $f(c) = 0$ if and only if the remainder in the synthetic division by $x - c$ is 0.

EXAMPLE 6 Show that -11 is a zero of the polynomial

$$f(x) = x^3 + 8x^2 - 29x + 44.$$

SOLUTION Dividing synthetically by $x - (-11) = x + 11$ gives us

$$\begin{array}{r|rrrr} -11 & 1 & 8 & -29 & 44 \\ & & -11 & 33 & -44 \\ \hline & 1 & -3 & 4 & 0 \end{array}$$

Thus, $f(-11) = 0$. ■

Example 6 shows that the number -11 is a solution of the equation $x^3 + 8x^2 - 29x + 44 = 0$. In Section 4.5 we shall use synthetic division to find rational solutions of equations.

Exercises 4.3

In Exercises 1–6 find the quotient $q(x)$ and the remainder $r(x)$ if $f(x)$ is divided by $g(x)$.

1 $f(x) = x^4 + 3x^3 - 2x + 5$, $g(x) = x^2 + 2x - 4$

2 $f(x) = 4x^3 - x^2 + x - 3$, $g(x) = x^2 - 5x$

3 $f(x) = 5x^3 - 2x$, $g(x) = 2x^2 + 1$

4 $f(x) = 3x^4 - x^3 - x^2 + 3x + 4$, $g(x) = 2x^3 - x + 4$

5 $f(x) = 7x^3 - 5x + 2$, $g(x) = 2x^4 - 3x^2 + 9$

6 $f(x) = 10x - 4$, $g(x) = 8x^2 - 5x + 17$

In Exercises 7–16 use synthetic division to find the quotient and remainder assuming the first polynomial is divided by the second.

7 $2x^3 - 3x^2 + 4x - 5$, $x - 2$

8 $3x^3 - 4x^2 - x + 8$, $x + 4$

9 $x^3 - 8x - 5$, $x + 3$

10 $5x^3 - 6x^2 + 15$, $x - 4$

11 $3x^5 + 6x^2 + 7$, $x + 2$

12 $-2x^4 + 10x - 3$, $x - 3$

13 $4x^4 - 5x^2 + 1$, $x - \frac{1}{2}$

14 $9x^3 - 6x^2 + 3x - 4$, $x - \frac{1}{3}$

15 $x^n - 1$, $x - 1$ where n is any positive integer

16 $x^n + 1$, $x + 1$ where n is any positive integer

In Exercises 17–28 use the Remainder Theorem to find $f(c)$.

17 $f(x) = 2x^3 - x^2 - 5x + 3$, $c = 4$

18 $f(x) = 4x^3 - 3x^2 + 7x + 10$, $c = 3$

19 $f(x) = x^4 + 5x^3 - x^2 + 5$, $c = -2$

20 $f(x) = x^4 - 7x^2 + 2x - 8$, $c = -3$

21 $f(x) = x^6 - 3x^4 + 4$, $c = \sqrt{2}$

22 $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, $c = -1$

23 $f(x) = x^4 - 4x^3 + x^2 - 3x - 5$, $c = 2$

24 $f(x) = 0.3x^3 + 0.04x - 0.034$, $c = -0.2$

25 $f(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$, $c = 4$

26 $f(x) = 8x^5 - 3x^2 + 7$, $c = \frac{1}{2}$

27 $f(x) = x^2 + 3x - 5$, $c = 2 + \sqrt{3}$

28 $f(x) = x^3 - 3x^2 - 8$, $c = 1 + \sqrt{2}$

In Exercises 29–32 use synthetic division to show that c is a zero of $f(x)$.

29 $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4,\ c = -2$

30 $f(x) = 4x^3 - 9x^2 - 8x - 3,\ c = 3$

31 $f(x) = 4x^3 - 6x^2 + 8x - 3,\ c = \frac{1}{2}$

32 $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1,\ c = -\frac{1}{3}$

33 Determine k so that $f(x) = x^3 + kx^2 - kx + 10$ is divisible by $x + 3$.

34 Determine all values of k such that $f(x) = k^2x^3 - 4kx - 3$ is divisible by $x - 1$.

35 Use the Factor Theorem to show that $x - 2$ is a factor of f [illegible] $3x^3 - 2x^2 + 5x + 6$.

36 Show that $x + 2$ is a factor of $f(x) = x^{12} - 4096$.

37 Prove that $f(x) = 3x^4 + x^2 + 5$ has no factor of the form $x - c$ where c is a real number.

38 Find the remainder if the polynomial $3x^{100} + 5x^{85} - 4x^{38} + 2x^{17} - 6$ is divided by $x + 1$.

39 Use the Factor Theorem to prove that $x - y$ is a factor of $x^n - y^n$ for all positive integers n. Assuming n is even, show that $x + y$ is also a factor of $x^n - y^n$.

40 Assuming n is an odd positive integer, prove that $x + y$ is a factor of $x^n + y^n$.