EXERCISES

1. Find the area under the line $y = 3x$ between $x = 1$ and $x = 5$ by the method employed in the text. *Ans.* 36.
2. Find the area of Exercise 1 but use the largest y-value in each subinterval in place of the smallest y-value.
3. Find the area under the parabola $y = x^2$ from $x = 0$ to $x = 5$ by using the largest y-value in each subinterval. *Ans.* $\frac{125}{3}$.
4. Find the area under the curve $y = x^2$ from $x = 1$ to $x = 5$. *Ans.* $\frac{124}{3}$.
5. Compute approximately the area under $y = x^2$ from $x = 0$ to $x = 5$ by using 10 subintervals Δx to fill out the interval (0, 5) and by using the largest y-value in each subinterval.
6. Compute approximately the area under the curve $y = 1/(1 + x^2)$ for the interval $0 \leq x \leq 1$. Use $n = 10$ and the smallest y-value in each subinterval. *Ans.* 0.75.

3. The Definite Integral. The method of finding areas that we examined in Section 2 introduces several new concepts into the calculus, and we should obtain a clearer understanding of them before considering whether we can do anything more significant with the method.

There is, first, a new limit concept. Each approximation to the area under a curve is a sum of rectangles. We have denoted these sums, when the minimum y-value is used in each Δx, by

$$\underline{S}_1,\ \underline{S}_2,\ \underline{S}_3,\ \cdots,\ \underline{S}_n,\ \cdots .$$

This set of successive sums is called an *infinite sequence*. More generally, an infinite sequence is a set of numbers such that there is a first member, a second member, and in fact a member corresponding to each positive integer n. Thus the set is unending. What is of interest in sequences is the number which the members of the sequence approach, that is, the *limit* toward which the members tend. We denote the number which the members approach by the notation

$$\lim_{n\to\infty} \underline{S}_n .$$

If we consider any infinite sequence, we need not use the bar underneath, and we can write it as

$$S_1, S_2, S_3, \cdots, S_n, \cdots .$$

Thus if the sequence consists of the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots,$$

then $S_1 = 1$, $S_2 = \frac{1}{2}$, $S_3 = \frac{1}{3}, \cdots, S_n = 1/n$. It is evident in this case that

$$\lim_{n\to\infty} S_n = 0$$

because as n gets larger and larger, the quantity $1/n$ comes closer and closer to 0. Thus if $n = 10^6$, $1/n = 0.000{,}001$; if $n = 10^9$, $1/n = 0.000{,}000{,}001$; and so on.

If the sequence consists of the numbers

$$1,\ 1\frac{1}{2},\ 1\frac{1}{3},\ 1\frac{1}{4},\ \cdots,\ 1\frac{1}{n},\ \cdots,$$

it is evident that the limit is 1 because the additional quantity $1/n$ in the nth term approaches 0.

Not every sequence has a limit. Thus the members of the sequence

$$1,\ 4,\ 9,\ \cdots,\ n^2,\ \cdots$$

become larger and larger and increase beyond bound. They do not approach a definite number. We sometimes say that the nth term becomes infinite, but this means only that the successive terms increase and become larger than any number that one may name.

A sequence may not have a limit or, one says, may fail to converge even if the terms do not become infinite. For example, consider

$$1,\ 2,\ \frac{1}{2},\ 1\frac{1}{2},\ \frac{1}{4},\ 1\frac{1}{4},\ \frac{1}{8},\ 1\frac{1}{8},\ \cdots.$$

Here the odd-numbered terms approach 0 and the even-numbered terms approach 1. Because *all* the terms do not come closer and closer to one fixed number, the sequence does not have a limit.

It is desirable to distinguish between a sequence and a function. The function $y = x^2$ takes on values for *all* values of x in some domain. If, for example, we were interested in this function over the domain $x = 3$ to $x = 5$, the possible values of x would be the whole numbers, fractions, and irrational numbers between 3 and 5. On the other hand, if we have a sequence whose nth term is, say, n^2, then only the values of n^2 for $n = 1$, $n = 2$, $n = 3$, and so on are of interest. One can regard n as a variable and n^2 as a function, but only the values of n^2 for positive integral values of n matter.

The difference between a function and a sequence insofar as the values that x can take on in the former case and n in the latter is reflected in the corresponding limit concepts. For example,

$$\lim_{x \to 2} x^2$$

is the number that x^2 approaches as x takes on *all* values closer and closer to 2. On the other hand, when one considers, for example,

$$\lim_{n \to \infty} \frac{1}{n^2},$$

he is interested in the number that $1/n^2$ approaches as n takes on larger and larger positive integral values.

We shall say more about sequences in connection with the work on infinite series. At the present time some acquaintance with the notions of infinite sequence and limit of a sequence is sufficient, and we may return to the subject of the area under a curve.

EXERCISES

1. Write the first 5 terms of the sequence whose nth term is the following:

(a) $\frac{n}{3}$. (b) n^2. (c) $\frac{n+1}{n+2}$. (d) $\frac{n+3}{\sqrt{n}}$.

2. Write the nth term of the following sequences:

(a) $3, 2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \cdots$. *Ans.* $2 + (1/n)$.

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$. *Ans.* $1/2^n$.

(c) $2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, 2\frac{1}{16}, \cdots$.

(d) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots$.

(e) $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \cdots$.

(f) $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \cdots$. *Ans.* $1/n!$

3. Determine by inspection the limit, if there is one, of each of the following sequences. The symbol s_n denotes the nth term of the sequence.

(a) $s_n = \frac{1}{\sqrt{n}}$. *Ans.* 0.

(b) $s_n = \frac{n}{n+1}$. *Ans.* 1.

(c) $s_n = \frac{1/n}{1/(n+1)}$. *Ans.* 1.

(d) $s_n = (-1)^n \frac{n-1}{n+1}$. *Ans.* None.

(e) $s_n = \sqrt{n}$.

(f) $s_n = (-1)^n \frac{1}{n!}$.

(g) $s_n = 1^n$.

(h) $s_n = \frac{n^2+3n}{n+2}$. *Ans.* None.

(i) $s_n = \frac{2(n-1)}{n^2-1}$.

(j) $s_n = 5 + \frac{1}{n} + \frac{6}{n^2}$.

We can generalize somewhat our method of finding the area under a curve. When we discussed previously approximating the area under a curve by rectangles, we decided that in each subinterval Δx we would choose the smallest y-value or the largest y-value. With either choice the successive sums obtained by utilizing narrower and narrower rectangles approach the area under the curve. If, however, we should select in each subinterval *any* y-value (Fig. 9-9), then the sum of the rectangular areas should likewise approach the area under the curve. This fact is rather easy to establish. Suppose that we have subdivided the interval (a, b) into equal subintervals Δx. As before, we denote by m_i the smallest y-value in the ith subinterval and by M_i, the largest y-value. Now let y_i denote any y-value in the ith subinterval. Thereby we obtain three different sequences of sums whose nth terms are

$$\underline{S_n} = m_1\,\Delta x + m_2\,\Delta x + \cdots + m_n\,\Delta x,$$

$$S_n^* = y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x,$$

$$\overline{S_n} = M_1\,\Delta x + M_2\,\Delta x + \cdots + M_n\,\Delta x.$$

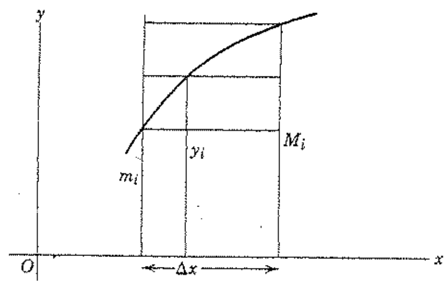


Figure 9-9

From the fact that

$$m_i \leq y_1 \leq M_i$$

we have that

$$\underline{S_n} < S_n^* < \overline{S_n}\,.$$

Now, when n becomes infinite, both $\underline{S_n}$ and $\overline{S_n}$ approach the area under the curve. Because S_n^* lies between the other two, it must approach the same limit. Hence, if we should find it convenient to choose some y-value in each subinterval Δx other than the minimum or maximum one, we may do so.

We may take another liberty with respect to the construction of the sequence of approximating sums. We introduced this concept by starting with subdivisions of the interval (a, b) into equal parts of width Δx. However, our main concern is to have the sum of the rectangles that we form come closer, as the number of rectangles increases, to the area under the curve. The way to attain this end is to make each rectangle narrower and narrower, even though in any one subdivision the rectangles are not of equal width. That the essential point is the narrowness of each rectangle is easily seen. Why is each rectangle $y_i\,\Delta x$ only an approximation to a portion of the area under the curve? An examination of Fig. 9-9 or of any of the foregoing figures in this chapter shows that the y_i we choose in any subinterval Δx is not necessarily the right choice because the y-values that correspond to the x-values in Δx vary and the y_i we choose may differ from the others. However, if Δx is small, the y-values corresponding to the x-values in Δx cannot differ very much from the y_i we choose, and the smaller Δx is, the less these y-values can differ from y_i. Hence what matters in forming the sequence of sums of rectangles is not that the Δx's in each subdivision of (a, b) be equal but that the Δx's approach 0 in size as n, the number of subintervals, increases. This requirement is usually stated thus: the maximum Δx in the nth subdivision must approach 0 as n becomes infinite. For if the maximum Δx approaches 0, so must each of the others.

If we do choose unequal subintervals to fill out the interval (a, b), we cannot denote the width of each subinterval by Δx. In place of this we use Δx_1, Δx_2, and so on for the successive widths (Fig. 9-10). If we let y_1 be the y-value corresponding to any x-value in Δx_1, let y_2 be the y-value corresponding to any x-value in Δx_2, and so on, the nth term of the sequence of

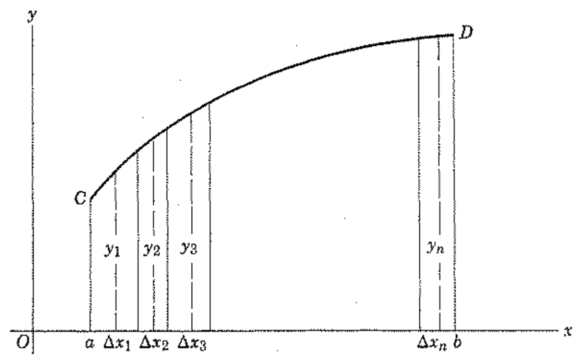


Figure 9-10

approximating sums is

$$S_n = y_1 \Delta x_1 + y_2 \Delta x_2 + \cdots + y_n \Delta x_n$$

and the area is given by

$$\lim_{n \to \infty} S_n, \tag{12}$$

provided that the maximum or largest Δx in S_n approaches 0 as n becomes infinite.

We see, then, that we can form many different kinds of sequences, each of which approaches the area under a curve, or, we may say, has the area under the curve as its limit. If we are given a particular curve and are to find the area between that curve and the x-axis and between two vertical lines at $x = a$ and $x = b$, we could choose any one of the sequences of approximating sums and then seek

$$\lim_{n \to \infty} S_n.$$

There is another notation for this limit which is very helpful in keeping before us the factors that determine the area. If $y = f(x)$ is the equation of the curve under which the area lies, then we write for the limit

$$\int_a^b y\,dx \quad \text{or} \quad \int_a^b f(x)\,dx. \tag{13}$$

That is, no matter which of the approximating sequences we may use,

$$\lim_{n \to \infty} S_n = \int_a^b y\,dx \quad \text{or} \quad \lim_{n \to \infty} S_n = \int_a^b f(x)\,dx.$$

The notation in (13) must not be taken too literally. The symbol $\int$ is the elongated S which we have already used to denote integration. It was introduced by Leibniz to remind one that he is dealing with the limit of a sequence of sums. The symbols a and b are the left- and right-hand end values of the x-domain over which the area is being calculated. The $y\,dx$ or $f(x)\,dx$ is a reminder that we took rectangles of height y_i and width Δx_i. If one is dealing with the specific function $y = x^2$ and wishes to indicate that function instead of y or $f(x)$, he can write

$$\int_a^b x^2\,dx. \tag{14}$$

Clearly the notation (13) or (14) is more informative than

$$\lim_{n\to\infty} S_n.$$

The quantity (13) is called the *definite integral*. The use of the word integral is not justified by what we have said so far, for integrals in the past have arisen through antidifferentiation. The connection between definite integrals and antidifferentiation remains to be discussed. The word "definite," however, is intended to convey the fact that the symbol (13) stands for a number, whereas ordinary integrals, or indefinite integrals as they are often called, are functions.

EXERCISES

1. Describe the area represented by the following:

(a) $\int_1^3 x^2\,dx$.

(b) $\int_0^5 x^3\,dx$.

(c) $\int_2^5 (x+3)\,dx$.

(d) $\int_{-1}^4 x^2\,dx$.

(e) $\int_1^3 (9 - x^2)\,dx$.

(f) $\int_3^8 (x-3)\,dx$.

(g) $\int_0^5 \sqrt{x}\,dx$.

2. The definite integral is (a) a sum, (b) a sequence of sums, (c) a limit of a sequence of sums, or (d) a limit of many sequences of sums. Which of the alternative answers is most appropriate?

3. Given that $S_n = x_1^3\,\Delta x_1 + x_2^3\,\Delta x_2 + \cdots + x_n^3\,\Delta x_n$, where $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$ fill out the x-interval (0, 1) and x_i is any value of x in Δx_i, express $\lim_{n\to\infty} S_n$ as a definite integral. *Ans.* $\int_0^1 x^3\,dx$.

4. Let $S_n = y_1\,\Delta x_1 + y_2\,\Delta x_2 + \cdots + y_n\,\Delta x_n$, where the Δx_i fill out the x-interval from $x = 1$ to $x = 5$ and y_i is a value of $3x^2$ in Δx_i. Express $\lim_{n\to\infty} S_n$ as a definite integral.

5. Let $S_n = 3x_1^2\,\Delta x + 3x_2^2\,\Delta x + \cdots + 3x_n^2\,\Delta x$, where the Δx are equal subintervals which fill out the x-interval from $x = -2$ to $x = 6$ and x_i is any value of x in each Δx. Express $\lim_{n\to\infty} S_n$ as a definite integral.

6. Let $S_n = x_1\sqrt{x_1^2 - 2}\,\Delta x_1 + x_2\sqrt{x_2^2 - 2}\,\Delta x_2 + \cdots + x_n\sqrt{x_n^2 - 2}\,\Delta x_n$, where the Δx_i fill out the x-interval from $x = 2$ to $x = 10$ and x_i is any value of x in Σx_i. Express $\lim_{n\to\infty} S_n$ as a definite integral.

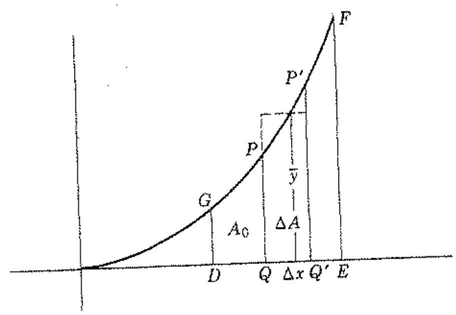


Figure 9-11

4. The Evaluation of Definite Integrals. Thus far in this chapter we have improved on the Greek method of finding areas bounded in whole or in part by curves by using sums of rectangles as the approximating figures and by using the equation of the curve and algebra to calculate the limit approached by the sequence of sums of rectangles. There is some question, however, as to whether we have gained very much. The few examples we have studied which show how to find an area by finding the limit of a sequence of sums also show that the process is cumbersome. Indeed, if the equation of the curve should be complicated, the summation technique might not be practical at all for the actual calculation. Is there an easy way of evaluating definite integrals?

Consider the area $DEFG$ (Fig. 9-11). We can regard this area as swept out by the line segment QP that starts at DG and moves to the right. Suppose that QP has reached the position shown. The position of QP is specified by the x-value of Q, say x_0, and the area generated for this position of QP, namely $DQPG$, can be denoted by A_0. Suppose that QP moves to $Q'P'$. Let the distance QQ' be denoted by Δx. When QP moves to $Q'P'$, the area increases by the amount $QQ'P'P$, which we denote by ΔA. This increment in area is larger than $QP \times \Delta x$ and smaller than $Q'P' \times \Delta x$. Because the ordinates between QP and $Q'P'$ increase continuously, there must be some ordinate between them, say $\bar{y}$, such that

$$\Delta A = \bar{y} \times \Delta x.$$

Then

$$\frac{\Delta A}{\Delta x} = \bar{y}.$$

To obtain the instantaneous rate of change of area, that is, the rate of change of A with respect to x at the value x_0 of x, we must find the limit of $\Delta A/\Delta x$ as Δx approaches 0. As Δx approaches 0, Q' moves to Q and $Q'P'$ moves to QP. For any value of Δx the ordinate $\bar{y}$ is always between QP and $Q'P'$. Hence $\bar{y}$ must also approach QP. The value of QP is the ordinate or y-value corresponding to x_0; that is, QP is y_0. Hence

$$\frac{dA}{dx} = y_0$$

or, if the equation of the curve is $y = x^2$,

$$\frac{dA}{dx} = x_0^2.$$

Because this result is true for any value of x in the interval DE, we may as well write the derived function, namely,

$$\frac{dA}{dx} = x^2.$$

To find the area $A(x)$ itself we apply antidifferentiation. Then

$$A(x) = \int x^2\,dx = \frac{x^3}{3} + C. \tag{15}$$

The problem of determining C arises. Here when QP is at DG, the value of A is 0. To be more specific, suppose that the x-value of D is 3. Then we know that when $x = 3$, $A = 0$. If we substitute these values in (15), we obtain

$$0 = \frac{3^3}{3} + C$$

or

$$C = -9.$$

Then

$$A(x) = \frac{x^3}{3} - 9$$

is the function which expresses the area from DG to any position of QP, the abscissa of Q being x.

To find the area $DEFG$, which we originally set out to do, we have but to note that this area is attained when QP reaches EF. Suppose that the abscissa of E is 6. Then we merely substitute 6 for x in the expression for A and obtain

$$A = \frac{6^3}{3} - 9 = 72 - 9 = 63.$$

Thus we have found the area bounded at least in part by a curve through the process of antidifferentiation or integration. To apply this process, we must of course know the equation of the curve.

We can obtain the same result if we take the expression (15) for the area, namely $(x^3/3) + C$, substitute 6 for x, then substitute 3 for x, and subtract the second result from the first. Thus

$$\frac{6^3}{3} + C - \left(\frac{3^3}{3} + C\right) = 63.$$

The constant of integration is eliminated in the process.

This area, which we obtained by antidifferentiation, is precisely the area we have already indicated by the symbol

$$\int_3^6 x^2\,dx. \tag{16}$$

Thus the definite integral (16) can be evaluated by antidifferentiation and the substitution of the end values 6 and 3 as indicated just above.

The result—that the definite integral, which is a limit of sequences of sums of rectangles, is evaluated essentially by antidifferentiation—is fundamental. It is, in fact, called the *fundamental theorem of the calculus*. We signalize it by stating it separately as a theorem:

Theorem: The definite integral $\int_a^b y\,dx$ or $\int_a^b f(x)dx$, which is a limit of sequences of sums of rectangles, is evaluated by finding the indefinite integral of the function y or $f(x)$ and by subtracting the result of substituting a in this integral from the result of substituting b in this integral.

In symbols we may state the fundamental theorem thus:

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a). \tag{17}$$

In this symbolism $F(x)$ is any antiderivative of $f(x)$; the symbol $F(x)|_a^b$ denotes that we intend to substitute b in $F(x)$, substitute a in $F(x)$ and subtract the second result from the first; the symbols $F(b) - F(a)$ indicate just what we get by carrying out what $F(x)|_a^b$ calls for.

We have stated a theorem but what have we done about its proof? We gave an argument based on the geometry of Fig. 9-11. However, the function represented geometrically there does not have the behavior of all functions. Hence the argument is incomplete. However, let us use this evidence as our "proof." It is intuitively sound. We shall examine a rigorous proof at a later time.

The definite integral, or the integral as the limit of sequences of sums, is a concept independent of the derivative, and it has been customary in the literature to think of the calculus as consisting of two parts: the differential calculus, concerned with differentiation and antidifferentiation, and the integral calculus, concerned with the definite integral and its ramifications. However, the fundamental theorem shows us that there is only one calculus. Nevertheless the definite integral is indeed something new and will prove to be a more important concept for tackling problems.

Let us consider an example of how the fundamental theorem is used.

Example. Find the area (Fig. 9-12) between the curve $y = x^2 + x + 1$, the x-axis, and the ordinates at $x = 2$ and $x = 4$.

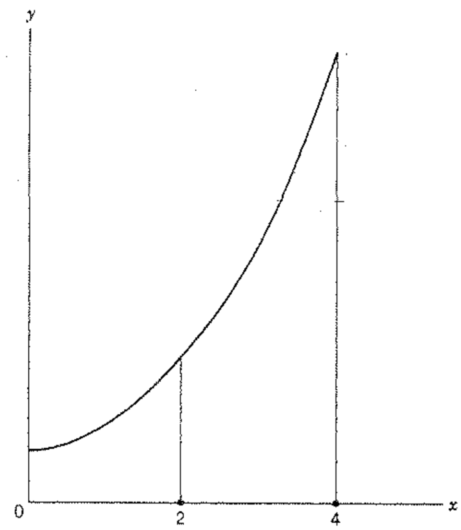


Figure 9-12

Solution. This area is represented by the definite integral

$$A = \int_2^4 (x^2 + x + 1)dx.$$

According to the fundamental theorem, we may evaluate the area by first finding an indefinite integral of

$$y = x^2 + x + 1.$$

One indefinite integral is

$$F(x) = \frac{x^3}{3} + \frac{x^2}{2} + x.$$

We ignore the constant of integration because this will drop out in the next step. The fundamental theorem tells us next that

$$\begin{aligned} A = F(x)\Big|_a^b &= \frac{x^3}{3} + \frac{x^2}{2} + x\Big|_2^4 \\ &= \frac{4^3}{3} + \frac{4^2}{2} + 4 - \left(\frac{2^3}{3} + \frac{2^2}{2} + 2\right) \\ &= \frac{80}{3} \end{aligned}$$

EXERCISES

1. Compute the following by using the fundamental theorem.

(a) $\int_1^3 x^2\,dx.$ *Ans.* $\frac{26}{3}$. (b) $\int_0^1 y\,dx$ where $y = x^3$. *Ans.* $\frac{1}{4}$.

(c) $\int_1^2 \frac{1}{x^2}\,dx.$

(d) $\int_{-3}^2 x^2\,dx.$

(e) $\int_0^5 3x^3\,dx.$

(f) $\int_0^{10} \sqrt{5x}\,dx.$

(g) $\int_1^5 (3x^2 - 2x + 5)dx.$

(h) $\int_1^5 \sqrt{4x+1}\,dx.$

(i) $\int_1^4 (5x - x^2)dx.$

2. Find the area bounded by the curve $y = x^2$, the x-axis, and the following:
(a) The ordinates at $x = 2$ and $x = 6$. *Ans.* $69\frac{1}{3}$.
(b) The ordinates at $x = 4$ and $x = 8$.

3. By the method of the calculus find the area bounded by the straight line $y = x$, the x-axis, and the ordinates at $x = 4$ and $x = 6$. Check your result by using plane geometry. *Ans.* 10.

4. Find the area bounded by the curve $y = 9x$, the x-axis, and the ordinates at $x = 3$ and $x = 6$.

5. Find the area bounded by the curve $y = x^{1/3}$, the x-axis, and the ordinates at $x = 2$ and $x = 8$. *Ans.* $12 - \frac{3}{2}\sqrt[3]{2}$.

6. Find the area bounded by the curve $y = x^2$, the x-axis, and the ordinate at $x = 5$.

7. Find the area between the curve $y = \sqrt{x+1}$, the x-axis, and the ordinates at $x = 1$ and $x = 5$. *Ans.* $\frac{2}{3}(6^{3/2} - 2^{3/2})$.

8. Given that $S_n = x_1^3\,\Delta x_1 + x_2^3\,\Delta x_2 + \cdots + x_n^3\,\Delta x_n$, where $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ fill out the interval (0, 1) and x_i is a value of x in Δx_i, show that, provided the maximum Δx_i of any subdivision of (0, 1) approaches 0 as n becomes infinite, $\lim_{n\to\infty} S_n = \frac{1}{4}$.

Suggestion: Express the limit of S_n as a definite integral and then use the fundamental theorem.

9. Let $S_n = y_1\,\Delta x_1 + y_2\,\Delta x_2 + \cdots + y_n\,\Delta x_n$, where the Δx_i fill out the x-interval from $x = 1$ to $x = 5$ and y_i is a value of $y = 3x^2$ in Δx_i. If the maximum Δx_i approaches 0 as n becomes infinite, evaluate $\lim_{n\to\infty} S_n$. *Ans.* 124.

10. Let $S_n = 3x_1^2\,\Delta x_1 + 3x_2^2\,\Delta x_2 + \cdots + 3x_n^2\,\Delta x_n$, where the Δx_i fill out the x-interval from $x = -2$ to $x = 6$ and x_i is any value of x in Δx_i. Evaluate $\lim_{n\to\infty} S_n$ with the understanding that the maximum Δx_i approaches 0 as n becomes infinite.

11. Let $S_n = 2x_1^2\,\Delta x + 2x_2^2\,\Delta x + \cdots + 2x_n^2\,\Delta x$, where the Δx are equal subintervals that fill out the x-interval from $x = 3$ to $x = 6$ and x_i is a value of x in the ith subinterval Δx. If $\Delta x = (6 - 3)/n$, evaluate $\lim_{n\to\infty} S_n$. *Ans.* 126.

12. Let $S_n = x_1\sqrt{x_1^2 - 2}\;\Delta x_1 + x_2\sqrt{x_2^2 - 2}\;\Delta x_2 + \cdots + x_n\sqrt{x_n^2 - 2}\;\Delta x_n$, where the Δx_i fill out the x-interval from $x = 2$ to $x = 10$ and x_i is any value of x in Δx_i. Assuming that the maximum Δx_i approaches 0 as n becomes infinite, evaluate $\lim_{n\to\infty} S_n$.

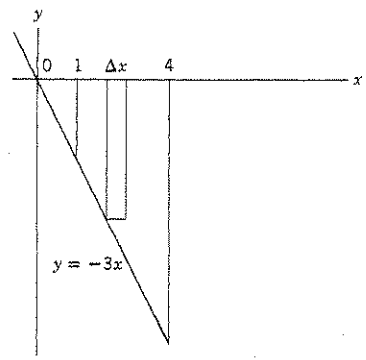


Figure 9-13

5. Areas Below the x-Axis. The areas we have considered so far were situated above the x-axis. Let us now consider the area illustrated in Figure 9-13, that is, the area between the x-axis, $y = -3x$, $x = 1$, and $x = 4$. Were we to approach this area by considering first a sequence of sums of rectangles, the nth term of which is

$$S_n = y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x,$$

everything we said previously about such sequences would hold except that the y_i would be negative. Hence $\lim_{n\to\infty} S_n$ would be a negative number.

As for the fundamental theorem, the argument we gave above as to the justification of the theorem applies here too. The fact that the y-values are negative does not in any way affect the argument because negative numbers are as respectable and as legitimate as positive numbers. In the present example

$$\lim_{n\to\infty} S_n = \int_1^4 -3x\,dx.$$

According to the fundamental theorem

$$\int_1^4 -3x\,dx = -\tfrac{3}{2}x^2\Big|_1^4 = -\tfrac{3}{2}\cdot 4^2 - \left(-\tfrac{3}{2}\cdot 1^2\right) = -22\tfrac{1}{2}.$$

What significance should we attach to the fact that the area is negative? The area itself as a purely geometrical quantity or physical quantity is positive. The fact that our method of representing curves by equations in which y or x or both can take on negative values leads, in the case where the area lies below the x-axis, to a negative area may be unfortunate but if we recognize this fact and take it into account where relevant, it does not cause any difficulty.

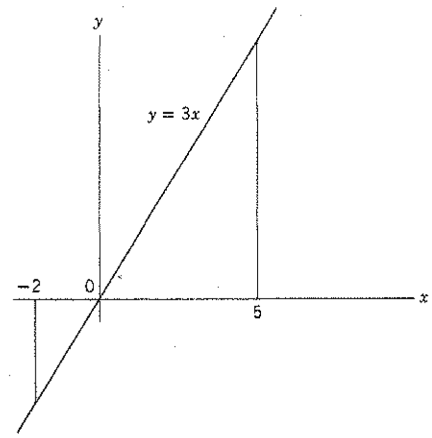


Figure 9-14

To see how we can handle this peculiar fact that areas lying below the x-axis turn out to be negative let us consider another example. Suppose we now seek the area (Fig. 9-14) bounded by the line $y = 3x$, the x-axis, and the ordinates at $x = -2$ and $x = 5$. If we evaluate

(18) $$\int_{-2}^{5} 3x\,dx$$

the result will not be the correct area because the integral is negative in part of the domain from -2 to 5 and positive in the other part. However, since the integral is a limit of a sequence of sums of rectangles we can consider the rectangles that "fill out" the area from -2 to 0 and the rectangles that fill out the area from 0 to 5 and consider the integral (18) as the sum of two integrals thus:

(19) $$\int_{-2}^{5} 3x\,dx = \int_{-2}^{0} 3x\,dx + \int_{0}^{5} 3x\,dx.$$

If we now apply the fundamental theorem to all of these integrals we obtain

$$\tfrac{3}{2}x^2\Big|_{-2}^{5} = \tfrac{3}{2}x^2\Big|_{-2}^{0} + \tfrac{3}{2}x^2\Big|_{0}^{5}$$

or

$$\tfrac{63}{2} = -6 + \tfrac{75}{2}.$$

The value of the definite integral on the left side of (19) is $\frac{63}{2}$. However, if we are interested in the *geometrical* area between $y = 3x$, the x-axis, $x = -2$, and $x = 5$ then, knowing that the integral gives a negative area when the area lies below the x-axis, we ignore the minus sign in the -6 and take the entire geometrical area to be $6 + \frac{75}{2}$ or $\frac{87}{2}$.

EXERCISES

1. Find the geometrical area between $y = -3x$, the x-axis, $x = 0$, and $x = 5$.

2. (a) Evaluate $\int_{-3}^{4} 3x\, dx$.
   (b) Find the geometrical area between $y = 3x$, the x-axis, $x = -3$, and $x = 4$.

3. (a) Evaluate $\int_{-3}^{3} -x^2\, dx$.
   (b) Find the geometrical area between $y = -x^2$, the x-axis, $x = -3$, and $x = 3$.

4. Find the geometrical area between $y = -\sqrt{2x+1}$, the x-axis, $x = 1$, and $x = 5$.

5. Find the geometrical area between the curve of $y = (x-3)(x-2)(x+1)$, the x-axis, $x = 0$, and $x = 4$. *Ans.* $\frac{71}{6}$.

6. Areas Between Curves. Our use of the definite integral and the fundamental theorem to find areas has been confined thus far to areas lying between a curve and the x-axis. Actually our new tools enable us to solve more complicated area problems.

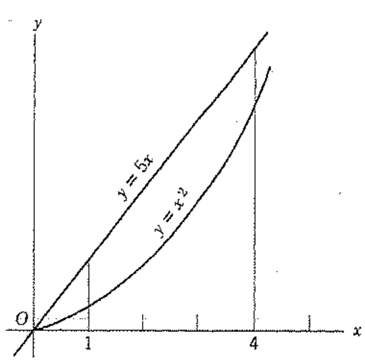
Suppose that we wished to find the area between the curves $y = 5x$, $y = x^2$ and the ordinates at $x = 1$ and $x = 4$ (Fig. 9-15). Clearly this area is the difference of two areas, the area under $y = 5x$ and the area under $y = x^2$ both taken between $x = 1$ and $x = 4$. Then since each area is given by a definite integral the area we seek is given by

(20) $$\int_1^4 5x\, dx - \int_1^4 x^2\, dx.$$

We can now use the fundamental theorem to evaluate each integral. Thus (20) yields

(21) $$\frac{5}{2}x^2\Big|_1^4 - \frac{x^3}{3}\Big|_1^4 = \frac{75}{2} - 21 = \frac{33}{2}.$$

Figure 9-15



However, in using the fundamental theorem we first find the indefinite integrals of $5x$ and of x^2, that is, we first find

$$\int 5x\,dx \qquad \text{and} \qquad \int x^2\,dx.$$

Since (20) calls for the subtraction of the two definite integrals we can use the fact (Chap. 6, Sect. 5) that the difference of the two indefinite integrals is the indefinite integral of the difference of the two functions, that is,

(22) $$\int 5x\,dx - \int x^2\,dx = \int (5x - x^2)dx$$

and since we then substitute the end values 1 and 4 in each of the separate integrals as in (21) we may as well substitute the end values 1 and 4 in the single integral on the right side of (22). Thus the area we seek is given by

$$\int_1^4 (5x - x^2)dx = \frac{5x^2}{2} - \frac{x^3}{3}\Big|_1^4 = \frac{5}{2}\cdot 4^2 - \frac{64}{3} - \left(\frac{5}{2}\cdot 1^2 - \frac{1}{3}\right) = \frac{33}{2}.$$

The point we have just made, stated in general terms, is that if u and v are functions of x then

(23) $$\int_a^b u\,dx \pm \int_a^b v\,dx = \int_a^b (u \pm v)dx.$$

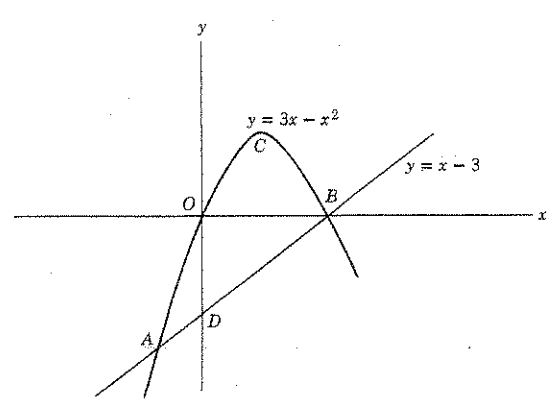
Sometimes the calculation of the right-hand integral in (23) is simpler because terms in u and v may combine or offset each other. Whether or not one uses (23) the main point of this section is that we can find the area between curves by means of the definite integral.

Let us consider another example.

Example. Find the area between the curves $y = 3x - x^2$ and $x - 3$.

Solution. Our first task is to recognize the area we wish to find. This is shown in Fig. 9-16. We must first find the abscissas of points A and B. This is done by

Figure 9-16



solving simultaneously

$$y = 3x - x^2$$
$$y = x - 3.$$

Then

$$3x - x^2 = x - 3$$

or

$$x^2 - 2x - 3 = 0.$$

Then $x = -1$ and $x = 3$. The value -1 is the abscissa of A and the value 3 is the abscissa of B. The respective ordinates are -4 and 0.

Now let us consider the area between O and B. Part of this area lies below the x-axis. We could find the area OBC and the area ODB and add the numerical values. However, if we form

$$\int_0^3 [(3x - x^2) - (x - 3)]dx$$

then because we have *subtracted* $x - 3$ we have changed the sign of the area ODB and the entire integral will give the correct geometrical area $OCBD$. In the region AOD, the y-values of $y = 3x - x^2$ are negative but the y-values of $y = x - 3$ are still more negative. Then

$$3x - x^2 - (x - 3)$$

will give precisely the vertical lengths between AO and AD and with the proper sign. Hence

$$\text{Area } AOD = \int_{-1}^0 [(3x - x^2) - (x - 3)]dx.$$

Then as in (19) the entire area $AOCBD$ is given by

$$\int_{-1}^3 [(3x - x^2) - (x - 3)]dx = \int_{-1}^3 (3 + 2x - x^2)dx$$
$$= 3x + x^2 - \frac{x^3}{3}\Big|_{-1}^3 = 10\tfrac{2}{3}.$$

EXERCISES

1. Evaluate $\int_1^4 (x^2 + x^3)dx$. *Ans.* $84\frac{3}{4}$.
2. Express the area under the curve of $y = 9 - x^2$ between $x = 0$ and $x = 1$ as a definite integral and then calculate it. *Ans.* $8\frac{2}{3}$.

3. Express the area between $y = x^3 + 9$, $y = x^2$, $x = 1$, and $x = 5$ as a definite integral and then evaluate it. *Ans.* $150\frac{2}{3}$.
4. Find the area between the curves $y = 3x^2$ and $y = 5x^2$ and between the ordinates at $x = 2$ and $x = 4$. *Ans.* $37\frac{1}{3}$.
5. Find the area between the curve $y = 1/x^2$ and the x-axis and between $x = 1$ and $x = 3$.
6. Find the area between the curve $y = x^2$ and the line $y = 2x$. *Ans.* $\frac{4}{3}$.
7. Find the area between the curves $y = x^2$ and $y = \sqrt{5x}$.
8. Find the area between the curves $y = 9 - x^2$ and $y = x^2$. *Ans.* $18\sqrt{2}$.
9. Find the area between the curve $y = x^2$ and the straight line $y = 8x - 4$.
10. Find the area between the parabolas $y = 2x^2 + 1$ and $y = x^2 + 5$. *Ans.* $10\frac{2}{3}$.
11. Find the area between $y^2 = 16x$ and $y^2 = x^3$. *Ans.* $8\frac{8}{15}$.
12. Calculate the physical area in the region bounded by the curve $y = x^3 - 8$, the x-axis, and the vertical lines $x = 1$ and $x = 3$. *Ans.* $12\frac{1}{2}$.
13. Prove with the aid of the fundamental theorem that
$$\int_a^b cy\,dx = c\int_a^b y\,dx.$$
14. Show by using a counterexample that $\int_a^b u\,dx - \int_a^b v\,dx \neq \int_a^b uv\,dx$ where u and v are functions of x.
15. By appealing to geometric evidence show that $\int_0^8 x^n\,dx + \int_0^1 x^{1/n}\,dx = 1$ for n a positive integer.
16. Show that the area under $y = x^a$, $a \neq -1$, and between $x = c$ and $x = d$ equals $1/a$ times the area bounded by $y = x^a$, the y-axis, $y = c^a$, and $y = d^a$.

7. Some Additional Properties of the Definite Integral. There are a few simple properties of the definite integral that are frequently used. The definition of the definite integral

$$\int_a^b y\,dx \tag{24}$$

assumes that the upper end value b is larger than the lower end value a. There is, however, no objection to considering the definite integral

$$\int_b^a y\,dx \tag{25}$$

where, with $a < b$, the upper end value is smaller than the lower one. We can, in fact, take over everything that applied to (24) with one exception. Previously, when we used equal subintervals in the interval (a, b), we took Δx to be $(b - a)/n$. For the sake of consistency we agree in the case of (25) that

$$\Delta x = \frac{a - b}{n}.$$

When $a < b$—the case that we are now discussing—Δx is negative. Moreover, even if we choose unequal subintervals in the interval (a, b), we take them to be negative. The choice of y-values in each subinterval can be the same for both (24) and (25). Hence the effect of our choice of sign for Δx is to make each S_n that we form for (25) the negative of the corresponding S_n for (24). However, if the sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, \frac{n-1}{n}, \ldots$$

approaches 1, the sequence

$$-\frac{1}{2}, -\frac{3}{4}, -\frac{7}{8}, \ldots, -\frac{n-1}{n}, \ldots$$

approaches -1. Hence

(26) $$\int_b^a y\,dx = -\int_a^b y\,dx.$$

We can consider the same fact from the standpoint of antidifferentiation. The left-hand integral calls for finding the indefinite integral of the function represented by y and then subtracting the result of substituting b in this indefinite integral from the result of substituting a. The right-hand integral calls for the same indefinite integral and then subtracting the result of substituting a from the result of substituting b. Then the final numbers will be the negatives of each other.

There is one more fact about the definite integral which is occasionally useful. Instead of considering the definite integral $\int_a^b f(x)dx$ we could consider

$$\int_a^x f(x)dx$$

wherein the upper end value x is variable; this is still called the definite integral. Of course now the value of the definite integral depends on the value of x; that is, it is a function of x. One can therefore ask, what is

(27) $$\frac{d}{dx}\int_a^x f(x)dx?$$

The notation just used, although perhaps understandable, is not quite satisfactory. The symbol x is used in two different senses. In $f(x)$, x stands for a variable which runs through some interval of values from a on. The symbol x at the upper end of the integral sign stands for the end of the interval of integration. To remove the ambiguity it is better to write

(28) $$\frac{d}{dx}\int_a^x f(u)du$$

wherein u runs through the values from a to x. Thus if $f(x)$ is x^2, $f(u)$ is u^2 and the integral in (28) is taken over the interval from a to x. Our question now is, what is the value of (28)?

The answer is readily obtained. By the fundamental theorem

$$\int_a^x f(u)du = F(x) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.* Then

$$\frac{d}{dx}\int_a^x f(u)du = f(x)$$

because the derivative of $F(x)$ is $f(x)$ and $F(a)$ is a constant. This result is often labeled the *corollary to the fundamental theorem*. Given the definite integral $\int_a^x f(u)du$, wherein the upper end value x is variable, then

(29)
$$\frac{d}{dx}\int_a^x f(u)du = f(x).$$

This corollary is occasionally useful.

Our treatment of the definite integral has been motivated by the problem of finding areas bounded by curves. We shall see later that the definite integral has many applications.

EXERCISES

1. Evaluate $\frac{d}{dx}\int_a^b x^3\,dx$ when a and b are constants.
2. Evaluate $\frac{d}{dx}\int_a^x u^3\,du$ by using the corollary to the fundamental theorem and by actually evaluating the integral.
3. If $g(x) = \int_0^x \sqrt{u^2 + 2}\,du$, what is d^2g/dx^2?
4. If $g(x) = \int_0^{x^2} f(u)du$, what is dg/dx?

   Suggestion: Let $x^2 = v$ and use the chain rule.
5. Criticize the following argument which "proves" that every triangle is isosceles. Consider triangle ABC (Fig. 9-17) and let AD be the altitude from A to BC. Now let PQ be any parallel to BC and PR and QS parallel to AD. Then $PR = QS$. By drawing parallels such as PQ we can cover triangle BAD by lines such as PR. Similarly, we can cover triangle CAD by lines such as QS. Hence triangle BAD equals CAD and triangle ABC is isosceles.

* One could also say, where $F(u)$ is an antiderivative of $f(u)$.

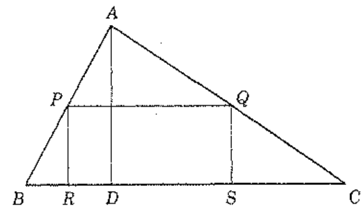


Figure 9-17

8. Numerical Methods for Evaluating Definite Integrals. By virtue of the fundamental theorem of the calculus we know that we can evaluate the definite integral

$$\int_a^b f(x)\,dx \tag{30}$$

by finding an antiderivative of $f(x)$, say $F(x)$, and then calculating $F(b) - F(a)$. For the functions we have studied thus far the problem of finding an antiderivative of $f(x)$ is readily solved and we can calculate the definite integral. However, we intend to apply the concept of the definite integral to many more complicated functions and even though we shall find that there are many more techniques of finding antiderivatives (Chap. 14), it is a sad fact that we cannot find antiderivatives for all of the $f(x)$ that occur in mathematical and physical problems even when $f(x)$ is an elementary function. For example, one cannot evaluate

$$\int_0^{\pi} \sqrt{1 + \cos^2 x}\;dx,$$

which gives the length of one arch of the sine curve (Fig. 10-1) by finding an antiderivative of the integrand. Moreover, in practical work some functions are known only as graphs or as statistical tables and even though it may be possible to find a formula to represent such functions the formula will surely be an approximate one and therefore, even if one can antidifferentiate the formula, the answer will still be approximate.

In both situations, that is, where one cannot find an antiderivative or where the antiderivative is an approximation, it is useful to be able to evaluate the definite integral numerically using only numerical values of $f(x)$.

One method of numerical evaluation is called the trapezoidal rule and this is readily derived. We know from our work on the definite integral (Sect. 3) that an approximate value of $\int_a^b f(x)dx$ is given by

$$S_n = y_0\,\Delta x_1 + y_1\,\Delta x_2 + \cdots + y_{n-1}\,\Delta x_n$$

where the subintervals $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ fill out the interval (a, b) and y_i is any value of $y = f(x)$ in Δx_i. Let us use equal subintervals and call each h.

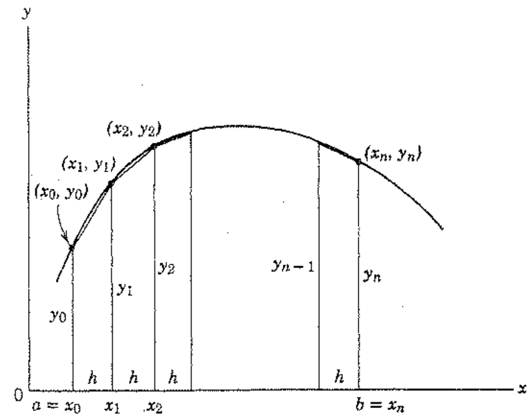


Figure 9-18

Then if we choose the y_i to be the left-hand y-values in each subinterval h (Fig. 9-18),

$$\underline{S}_n = y_0h + y_1h + \cdots + y_{n-1}h$$

is an approximation to the definite integral (30). It is also true that

$$\overline{S}_n = y_1h + y_2h + \cdots + y_nh,$$

where the y_i are the right-hand y-values in each subinterval h, is an approximation to the definite integral (30). Hence the average of the two should be a better approximation. This average is

(31) $$S_n = \tfrac{1}{2}(y_0 + y_1)h + \tfrac{1}{2}(y_1 + y_2)h + \cdots + \tfrac{1}{2}(y_{n-1} + y_n)h.$$

Each of these terms is the area of a trapezoid formed by, for example, y_0, y_1, h and the chord joining (x_0, y_0) and (x_1, y_1), because the area of a trapezoid is one-half the altitude times the sum of the bases. Hence the approximation (31) is called the trapezoidal rule. We can rewrite (31) in the more convenient form

(32) $$\int_a^b f(x)dx \approx h\left[\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n\right]$$

wherein the symbol $\approx$ means approximately. Generally, the larger n is, the better the approximation.

Example. Approximate the area under the curve $y = 1/(1 + x^4)$ from $x = 0$ to $x = 2$ (Fig. 9-19).

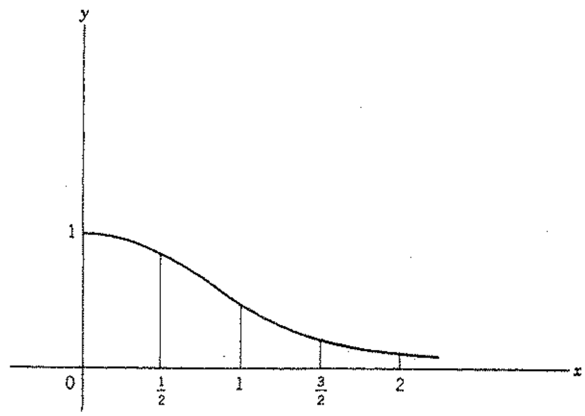


Figure 9-19

Solution. Let us divide up the interval from $x = 0$ to $x = 2$ into 4 subintervals. Then $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, and $x_4 = 2$, while $h = \frac{1}{2}$.

By actual substitution in the function $y = 1/(1 + x^4)$ we find that

$$y_0 = f(x_0) = f(0) = 1; \quad y_1 = f(x_1) = f(\tfrac{1}{2}) = .94;$$

$$y_2 = f(x_2) = f(1) = .5; \quad y_3 = f(x_3) = f(\tfrac{3}{2}) = .165;$$

$$y_4 = f(x_4) = f(2) = .059.$$

Substitution in (32) yields

$$\tfrac{1}{2}\left[\tfrac{1}{2}(1) + .94 + .5 + .165 + \tfrac{1}{2}(.059)\right] = 1.067.$$

EXERCISES

1. Evaluate $\int_1^5 x^2\,dx$ approximately by using the trapezoidal rule and 4 subintervals. Then determine the accuracy of the approximation by using the fundamental theorem.
2. Approximate $\int_0^{\frac{1}{2}} \frac{dx}{1 + x^2}$ by using five subintervals. *Ans.* 0.463.
3. Approximate $\int_0^3 \sqrt{1 + x^3}\,dx$ using six subintervals. *Ans.* 7.39.
4. Suppose a function $y = f(x)$ is known to us only through the following table:

x	0	1	2	3	4	5	6	7	8	9	10
y	1.72	1.60	1.44	1.24	1.06	0.92	0.80	0.70	0.63	0.56	0.50

Approximate $\int_0^{10} f(x)dx$.

The trapezoidal rule approximates a curve by a set of line segments or chords and in effect uses the formula for the area of a trapezoid. One would expect to do better by approximating the given curve by another curve which is readily integrated or whose values are readily computed. The simplest approximating curve which usually improves on the trapezoidal rule and which is easy to handle is a parabola of the form $y = ax^2 + bx + c$. Such a parabola is determined by three points. Hence we divide up the interval (a, b) of the definite integral $\int_a^b f(x)dx$ into n equal subintervals. Corresponding to the endpoints $x_0, x_1, x_2, \ldots, x_n$ of these subintervals there are the points $P_0, P_1, \ldots, P_n$ of the given curve (Fig. 9-20). We approximate the curve through P_0, P_1, and P_2 by an arc of a parabola; then do the same for P_2, P_3, and P_4, for P_4, P_5, and P_6, and so on. Since each parabolic arc covers two subintervals, we see that the number of these must be even.

Now let us investigate what the area under a typical parabolic arc is. The equation of the parabola is $y = ax^2 + bx + c$ and it is to be determined by three points on the curve, say (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}). We can come closer to our ultimate goal if we write the equation of the parabola as

(33) $$y = a(x - x_i)^2 + b(x - x_i) + y_i.$$

This form insures that the parabola goes through (x_i, y_i). Now since we choose equal subintervals of the width h, say, then the area we want is

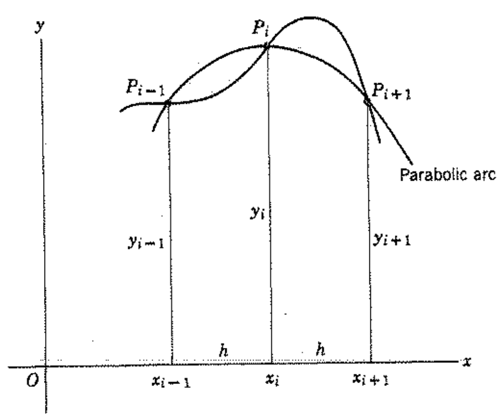
$$\int_{x_i-h}^{x_i+h} \left[a(x - x_i)^2 + b(x - x_i) + y_i\right] dx$$

$$= \left. \frac{a(x - x_i)^3}{3} + \frac{b(x - x_i)^2}{2} + y_i x \right|_{x_i-h}^{x_i+h}$$

$$= 2y_i h + \frac{2ah^3}{3}$$

(34) $$= h\left(2y_i + \frac{2ah^2}{3}\right).$$

Figure 9-20



To make the parabola pass through (x_{i-1}, y_{i-1}) we must have from (33), since $x - x_i$ must be $-h$,

$$y_{i-1} = ah^2 - bh + y_i;$$

and to make the parabola pass through (x_{i+1}, y_{i+1}) we must have

$$y_{i+1} = ah^2 + bh + y_i.$$

If we now add these last two equations and rearrange terms we get

$$2ah^2 = y_{i-1} - 2y_i + y_{i+1}.$$

Hence by (34) the area under the parabola through (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) is

(35) $$\frac{h}{3}(y_{i-1} + 4y_i + y_{i+1}).$$

We planned to divide the interval (a, b) into n subintervals, with n even, and to approximate each arc of the curve through P_0, P_1 and P_2, P_2, P_3 and P_4, and so on by an arc of a parabola. The areas under these parabolic arcs are given by (35) by letting $i = 1, 3, 5, \ldots, n - 1$. Then the approximation to the desired integral is given by adding these successive areas so that

(36) $$\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\right].$$

This result is known as Simpson's rule.

Example. Let us calculate $\int_1^2 \frac{dx}{x}$ and let us choose 10 subintervals. Then $h = 0.1$ and $y_0 = \frac{1}{1}$, $y_1 = 1/1.1$, $y_2 = 1/1.2, \ldots, y_{10} = \frac{1}{2}$. If we substitute these values in (36) we find that the definite integral is approximately 0.693150. We shall find later that the value of this integral is the logarithm of 2 to a base e which we have yet to discuss. A more accurate value of the integral computed by other means is 0.693147.

EXERCISES

1. Approximate $\int_0^1 \frac{dx}{1 + x}$ using 4 subintervals. *Ans.* 0.693.

2. Evaluate $\int_1^5 x^2\,dx$ by using Simpson's rule and 4 subintervals. Compare the result with the exact value obtained by using the fundamental theorem. Would you expect the two values to agree?

3. Approximate $\int_0^3 \sqrt{1 + x^3}\, dx$ by using Simpson's rule and 6 subintervals.

4. Given the following data on a function $y = f(x)$

x	0	1	2	3	4	5	6
y	32	38	29	33	42	44	38

calculate approximately $\int_0^6 f(x)dx$. *Ans.* 37.33.

5. Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{1 + x^2}$ approximately by using Simpson's rule and 4 subintervals.

APPENDIX
THE SUM OF THE SQUARES OF THE FIRST n INTEGERS

We wish to prove that

$$S = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Proof: One method of proof depends on a trick. We have the identity

$$n^3 - (n - 1)^3 = 3n^2 - 3n + 1.$$

By replacing n by $n - 1$, we have

$$(n - 1)^3 - (n - 2)^3 = 3(n - 1)^2 - 3(n - 1) + 1.$$

Similarly,

$$\begin{aligned}
(n - 2)^3 - (n - 3)^3 &= 3(n - 2)^2 - 3(n - 2) + 1. \\
&\cdots\cdots\cdots \\
3^3 - 2^3 &= 3 \cdot 3^2 - 3 \cdot 3 + 1. \\
2^3 - 1^3 &= 3 \cdot 2^2 - 3 \cdot 2 + 1. \\
1^3 - 0^3 &= 3 \cdot 1^2 - 3 \cdot 1 + 1.
\end{aligned}$$

If we now add the left sides and then the right sides, we have

$$\begin{aligned}
n^3 &= 3(1^2 + 2^2 + \cdots + n^2) - 3(1 + 2 + \cdots n) + n \\
&= 3S - 3\frac{n}{2}(n + 1) + n.
\end{aligned}$$

If we solve this equation for S, we obtain the result above.

The proof can also be made by mathematical induction.

CHAPTER TEN THE TRIGONOMETRIC FUNCTIONS

1. Introduction. We have learned the basic concepts of the calculus, differentiation, antidifferentiation, and the definite integral. We have also seen that these three concepts are intimately related. However, we have applied these concepts only to simple functions, that is, polynomials such as $x^2 - x + 5$, rational functions, which are quotients of polynomials, and to simple expressions involving fractional powers of x or algebraic functions of x as, for example, $\sqrt{(x + 5)/(x^2 + x)}$. Because we have been limited to just these few types of functions we have been limited in the applications we could make. To extend the power of the calculus we must learn how to handle new types of functions.

Our next concern will be the class known as trigonometric functions. These are important because they represent periodic phenomena. The motion of a bob on a spring and the motion of a pendulum are obvious periodic phenomena, but many others, such as sound waves, alternating electric current, and radio waves, are also periodic, although this fact is not at once apparent. However, even if one recognizes that a phenomenon is periodic and that trigonometric functions should be involved, he must still answer the question of precisely which function represents that phenomenon and how one can extract physical information from the function. After presenting a few technical facts about the trigonometric functions we shall consider a few of these applications.

2. The Sinusoidal Functions. The basic periodic function is $y = \sin x$. To refresh our minds about this function, let us look at the graph in Fig. 10-1. As the function arises in trigonometry, x stands for the size of an angle, which is measured in degrees or in radians. Both units are shown in the figure. In the calculus the radian measure is preferred for a reason that will be evident shortly.

There is a common misconception about the function $y = \sin x$ which stems from the fact the x-values originate as a measure of angles. The

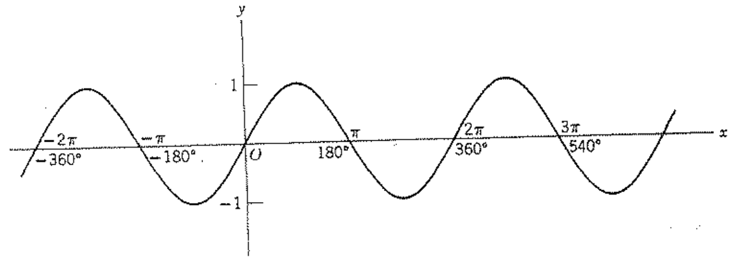


Figure 10-1

function $v = 32t$ originally represented and still represents the velocity of a body which is dropped and falls (in a vacuum) under the pull of the earth's gravity. The values of t in this application are time values and the v-values are, of course, velocity values. However, as a mathematical function the values of v and t are pure numbers and not time and velocity values. All one can say mathematically is that when $t = 1$, $v = 32$. Indeed, the same mathematical function might be used in a totally different physical context where t might represent any number of oxen and v the combined lengths of their tails. The same point applies to $y = \sin x$. As a mathematical function x and y represent pure numbers, and the fact that x originally represented the sizes of angles is irrelevant. If we choose to let the values of x represent, for example, values of time and the values of y to represent distance, we are entirely free to do so, and we shall do so when this interpretation fits the physical phenomenon under study. In other words, once we have some way of determining the y-value which belongs to a given x-value, we have a mathematical function that can be applied to any physical situation in which the function may be useful.

To obtain the value of y for a given value of x we do interpret x as the number of radians in an angle and then look up our trigonometric tables for the sine of the angle of that many radians or, if necessary, first convert the radians to degrees. This reversion to angles and sines of angles is utilized *only because the values of y happen to be recorded in trigonometric tables.*

The function $y = \sin x$ is periodic, that is, the y-values repeat in successive intervals the values that y takes on in the interval 0 to 2π. The interval 2π is called the *period*, a term which, incidentally, comes from the physical situation in which x represents time. (In applications in which x represents distance, the interval 2π is called the *wavelength*.) As Fig. 10-1 shows, the maximum y-value is 1, and this number is called the *amplitude* of the function.

The great usefulness of the trigonometric functions derives from the fact that for each of the six fundamental functions, $y = \sin x, y = \cos x, y = \tan x, y = \cot x, y = \sec x$, and $y = \csc x$, there is an infinitude of variations. We shall discuss these in connection with the function $y = \sin x$ but the same remarks apply to the other five functions.

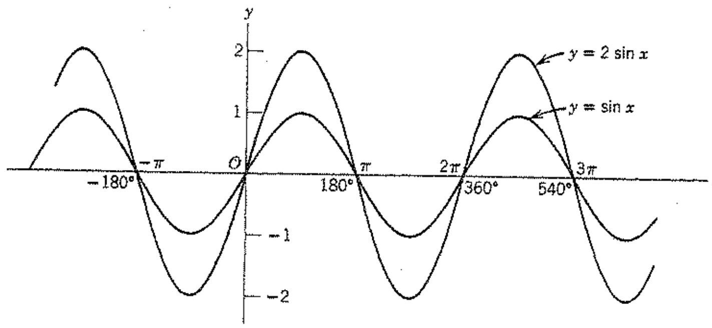


Figure 10-2

A common variation of $y = \sin x$ is

$$y = 2 \sin x. \tag{1}$$

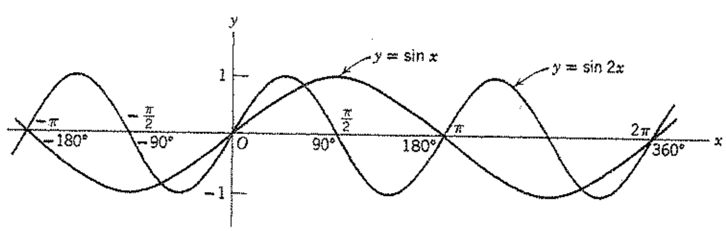
It is easy to see in this case how the second function differs from the first. For each value of x, $2 \sin x$ is twice $\sin x$. Figure 10-2 shows the effect of the factor 2. The period of $y = 2 \sin x$ is still 2π, but the amplitude is 2. Of course, any number can occur in place of the 2, and so we already have the infinite number of functions $y = a \sin x$.

Another equally common variation of $y = \sin x$ is exemplified by

$$y = \sin 2x. \tag{2}$$

The effect of the 2 in this function is different from that in $y = 2 \sin x$. In the case of (2), given any value of x, say $\pi/4$, we first multiply by 2, obtaining $\pi/2$, and then find $\sin \pi/2$, which is of course 1. That is, for $y = \sin 2x$ when $x = \pi/4$, $y = 1$. We can readily see that as x takes on the values from 0 to π, $2x$ takes on the values from 0 to 2π and $\sin 2x$ runs through the complete set of y-values that occur in the interval from 0 to 2π of $y = \sin x$. That is, y increases from 0 to 1, then decreases to 0, decreases still further to -1, and then increases to 0. This range of y-values is called a cycle. The behavior of $y = \sin 2x$ is represented in Fig. 10-3. We see that the amplitude is 1 but the period is π. It is worth remembering that the

Figure 10-3



period can be obtained by dividing the normal period 2π by the coefficient of x. It is perhaps unnecessary to add that in place of the 2 in $y = \sin 2x$, any number can occur, so that there is another infinity of sinusoidal functions of the form $y = \sin bx$.

Another kind of variation on $y = \sin x$ is that which combines the two types we have discussed. It is generally represented by the formula $y = a \sin bx$. A few other variations are suggested in the exercises.

EXERCISES

1. Sketch on the same axes $y = \sin x$ and
   (a) $y = \sin 3x$.
   (b) $y = 3 \sin x$.
   (c) $y = 3 \sin 2x$.
   (d) $y = 2 \sin 4x$.
   (e) $y = \sin\left(x + \frac{\pi}{2}\right)$.
   (f) $y = \sin x + \frac{\pi}{2}$.
   (g) $y = \sin(x - 1)$.
   (h) $y = -2 \sin 3x$.
2. Sketch $y = x + \sin x$.

   Suggestion: Sketch $y = x$ and $y = \sin x$ on the same axes. Then at a number of values of x add the ordinates of the two curves. If one ordinate is positive and the other negative, you must of course add the signed values.
3. Sketch $y = x \sin x$.
4. Sketch $y = 3 \sin 2(x - 1) + 4$.

   Suggestion: Sketch in order $y = \sin x$, $y = \sin 2x$, $y = \sin 2(x - 1)$, $y = 3 \sin 2(x - 1)$, and $y = 3 \sin 2(x - 1) + 4$.
5. Sketch $y = x^2 + \sin x$.

3. Some Preliminaries on Limits. In the calculus we are interested in what we can do with the derived functions and integrals of the trigonometric functions. To find the derivative of $y = \sin x$, we use the method of increments. In the course of this work we must determine certain limits and so we shall dispose of these now. The first is

$$\lim_{x \to 0} \frac{\sin x}{x} . \tag{3}$$

The determination of this limit is more difficult than that of, say,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} . \tag{4}$$

Here we can divide numerator and denominator by $x - 2$, which is certainly correct when $x \neq 2$; having obtained $x + 2$, we can see that the function approaches 4 as x approaches 2. For (3), however, where the same difficulty arises as in (4), namely, both numerator and denominator approach 0, it is not possible to divide numerator and denominator by some quantity in order to determine the limit readily. Moreover, substituting 0 for x in (3), which sometimes gives the same result as finding the limit as x approaches 0, does not help because in the present case it gives $0/0$.

Before trying to determine the limit of a function, it is wise to convince oneself that it does indeed have one. To obtain some indication of whether there may be a limit and what number to expect, we choose values of x and calculate the values of the fraction (3). Thereby we obtain the following table:

x	0.5	0.3	0.2	0.1	0.05	0.01
$\sin x$	0.479	0.2955	0.1987	0.0998	0.049979	0.00999998
$\frac{\sin x}{x}$	0.959	0.985	0.9933	0.9983	0.99958	0.999998

It seems quite clear that the limit in question is 1.

We shall prove that this is so. We may regard x as the size of a positive central angle in a circle (Fig. 10-4) of radius 1 and center O. Let AD be the arc of the circle intercepted by the angle. At A we drop the perpendicular AB to the side OD of the angle and at D we erect the perpendicular to the side OD. This perpendicular meets OA in some point which we denote by E.

We see from the figure that

(5) $$\text{area } OBA < \text{area of sector } ODA < \text{area } ODE.$$

But

$$\text{area } OBA = \tfrac{1}{2} OB \cdot BA = \tfrac{1}{2} \cos x \sin x.$$

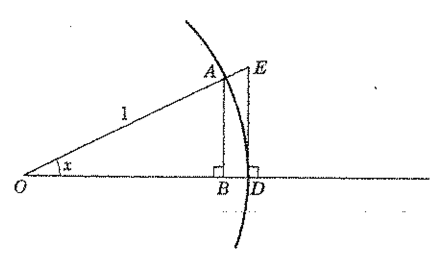
Further, the area ODA of the sector of the circle is that part of the entire area of the circle which the central angle x is of 2π. That is,

$$\text{area } ODA = \frac{x}{2\pi} \cdot \pi (1)^2 = \frac{x}{2}.$$

Finally,

$$\text{area } ODE = \tfrac{1}{2} ED \cdot OD = \tfrac{1}{2} \tan x = \frac{1}{2} \frac{\sin x}{\cos x}.$$

Figure 10-4



With these values of the individual areas the inequality (5) reads

$$\tfrac{1}{2}\cos x \sin x < \frac{x}{2} < \frac{1}{2}\,\frac{\sin x}{\cos x}.$$

If we divide the inequality by the positive quantity $(\sin x)/2$ we obtain

$$\textbf{(6)} \qquad \cos x < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

We recall from the study of trigonometry (see Fig. 10-5) that as x approaches 0, $\cos x$ approaches 1. However, we see from (6) that as x approaches 0, $x/\sin x$ always lies between two quantities, $\cos x$ which approaches 1 and $1/\cos x$ which must also approach 1 because $\cos x$ does. Hence

$$\lim_{x\to 0} \frac{x}{\sin x} = 1.$$

Because this limit is 1, it is also true that the reciprocal approaches 1, that is,

$$\textbf{(7)} \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We have considered the limit of $(\sin x)/x$ as x approaches 0 through positive values. However, when x is negative, we may use the fact that

$$\textbf{(8)} \qquad \frac{\sin x}{x} = \frac{\sin(-x)}{-x},$$

because $\sin(-x) = -\sin x$. Moreover, if x is negative, $-x$ is positive. Because the right side of (8) deals with a positive variable and the fraction approaches 1 as x approaches 0, the left side of (8) also approaches 1.

We should note that the convenience of radian measure is utilized in the proof of (7). Had we used x-values which are associated with the degree measure of angles, our statement about the area of sector ODA would have had to be modified to read

$$\text{area } ODA = \frac{x}{360}\cdot\pi(1)^2,$$

and as a consequence the final result would have been

$$\lim_{x\to 0} \frac{180}{\pi}\,\frac{\sin x}{x} = 1.$$

When the values of x are associated with the radian measure of angles, we dispense with the factor $180/\pi$. Of course, nothing important in the calculus would really be affected if we did have to carry this factor.

There is one more limit that we shall utilize shortly, namely,

$$\textbf{(9)} \qquad \lim_{x\to 0} \frac{1-\cos x}{x}.$$

The difficulty in evaluating this limit is the usual one—both numerator and denominator approach 0. However, the limit is readily evaluated. We utilize the trigonometric identity

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} \tag{10}$$

so that

$$1 - \cos x = 2\sin^2\frac{x}{2}.$$

If we substitute this value in (9), we obtain

$$\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x}.$$

It is correct algebraically to write this last expression as

$$\lim_{x\to 0}\sin\frac{x}{2}\,\frac{\sin\frac{x}{2}}{\frac{x}{2}}.$$

If, for convenience, we regard $x/2$ as y, we see that we must determine the limit as y approaches 0 of a product one factor of which is $\sin y$ and the other $(\sin y)/y$. We know that the limit of a product of two functions is the product of the limits. However, as y approaches 0, $(\sin y)/y$ approaches 1 and $\sin y$ approaches 0. Hence the product of the limits is 0. Thus

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0. \tag{11}$$

EXERCISES

1. Show that $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$ by multiplying numerator and denominator by $1 + \cos x$ and then making any appropriate steps.
2. Evaluate the following limits:
   (a) $\lim_{x\to 0}\frac{\sin 2x}{x}$. *Ans.* 2.
   (b) $\lim_{x\to 0}\frac{\sin ax}{x}$.
   (c) $\lim_{\Delta x\to 0}\frac{\sin \Delta x}{\Delta x}$. *Ans.* 1.
   (d) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.
   (e) $\lim_{x\to 0}\frac{\tan x}{x}$. *Ans.* 1.

4. Differentiation of the Trigonometric Functions. To differentiate $y = \sin x$ we apply the method of increments. Let x_0 be the value of x at which we desire the derivative. Then

$$y_0 = \sin x_0.$$

If x changes to the value $x_0 + \Delta x$, then

$$y_0 + \Delta y = \sin(x_0 + \Delta x)$$

and

$$\frac{\Delta y}{\Delta x} = \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x}.$$

We cannot obtain the limit of the right side as Δx approaches 0 by merely inspecting the expression; hence we shall try to transform it on the chance that some other form may be more perspicuous. The presence of $\sin(x_0 + \Delta x)$ suggests that we try to use the identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Then

$$\frac{\Delta y}{\Delta x} = \frac{\sin x_0 \cos \Delta x + \cos x_0 \sin \Delta x - \sin x_0}{\Delta x}.$$

Again, a direct evaluation of the limit of the fraction does not seem possible, and therefore we try collecting the terms in $\sin x_0$ and breaking up the fraction into two fractions. Thus

$$\textbf{(12)} \qquad \frac{\Delta y}{\Delta x} = \sin x_0 \cdot \frac{\cos \Delta_x - 1}{\Delta x} + \cos x_0 \frac{\sin \Delta x}{\Delta x}.$$

To obtain dy/dx we must determine the limit of $\Delta y/\Delta x$ as Δx approaches 0. The right side is a sum of two terms, and according to the theorem that the limit of a sum is the sum of the limits we may consider each term separately. As for the term

$$\textbf{(13)} \qquad \sin x_0 \cdot \frac{\cos \Delta x - 1}{\Delta x},$$

the quantity $\sin x_0$ is a constant. A theorem on limits tells us that the limit of the product is the constant times the limit of the second factor. This limit is precisely the one we considered in (11), except for a factor of -1 and the fact that the x in (11) is the Δx in (13). Hence the limit of (13) is 0.

The term

$$\textbf{(14)} \qquad \cos x_0 \frac{\sin \Delta x}{\Delta x}$$

in (12) is again a constant times a function of Δx, and the limit of this term is the constant times the limit of $(\sin \Delta x)/\Delta x$. By (7) we see that the limit of (14) is $\cos x_0$.

We have, then, from the consideration of (13) and (14) that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \cos x_0.$$

This result holds at every value of x_0, and therefore we have that if $y = \sin x$ the derived function is

$$\frac{dy}{dx} = \cos x. \tag{15}$$

Now that we have the derived function in (15), the chain rule enables us to handle more complicated sine functions. Suppose that $y = \sin 3x$. To obtain dy/dx we regard $3x$ as u; we then have

$$y = \sin u, \qquad u = 3x. \tag{16}$$

The chain rule states that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Hence from (15) we have

$$\frac{dy}{dx} = \cos u \cdot 3 = 3 \cos 3x.$$

In fact, no matter how complicated the function of x whose sine is being considered, we may always call it u and write $y = \sin u$ with u representing that function of x. Then

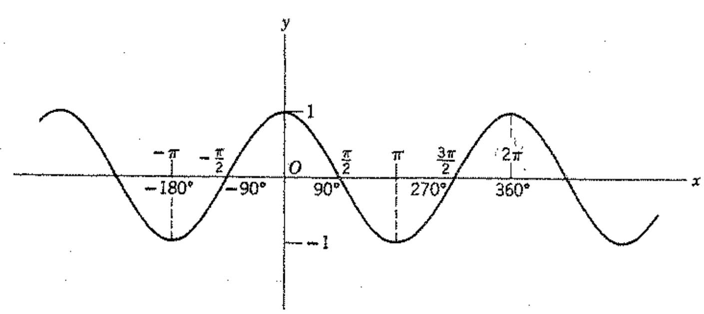
$$\frac{dy}{dx} = \cos u \frac{du}{dx}. \tag{17}$$

The derivatives of the other five trigonometric functions are readily obtained. Let us remind ourselves first of the behavior of the function $y = \cos x$. This is shown in Fig. 10-5.

To obtain the derivative of $y = \cos x$, we can go through the method of increments, but it is easier to obtain it from (17). We know that we can relate $\cos x$ to $\sin x$ through the trigonometric identity

$$\cos x \equiv \sin\left(x + \frac{\pi}{2}\right).$$

Figure 10-5



This identity holds for every value of x. Hence to differentiate $y = \cos x$, we may differentiate $y = \sin(x + \pi/2)$. With the chain rule the result is immediate. We regard $x + \pi/2$ as u, so that

$$y = \sin u, \qquad u = x + \frac{\pi}{2}.$$

Then by (17)

$$\frac{dy}{dx} = \cos u \cdot 1 = \cos\left(x + \frac{\pi}{2}\right).$$

But we have the identity

$$\cos\left(x + \frac{\pi}{2}\right) \equiv -\sin x.$$

Hence, if $y = \cos x$,

(18)
$$\frac{dy}{dx} = -\sin x.$$

As in the case of $y = \sin x$, if we have $y = \cos u$, where u is some function of x, we may apply the chain rule to write

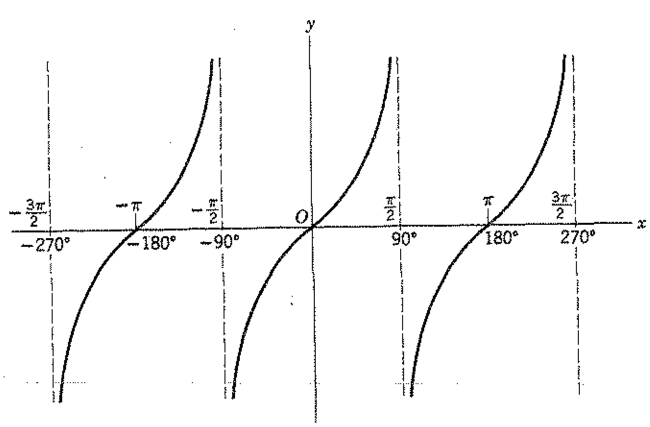
(19)
$$\frac{dy}{dx} = -\sin u \frac{du}{dx}.$$

For the function $y = \tan x$, whose behavior is shown in Fig. 10-6, the derived function is readily obtained. We have only to note that

$$y = \tan x = \frac{\sin x}{\cos x},$$

and we may now apply the theorem on the derivative of the quotient of two functions. We leave the details for an exercise and merely note the result. If

Figure 10-6



$y = \tan x$, then

(20) $$\frac{dy}{dx} = \sec^2 x.$$

Here we should note that the derived function exists at all values of x, except when x is an odd multiple of $\pi/2$ (see Fig. 10-8). As in the case of $y = \sin x$, if we have $y = \tan u$, where u is a function of x, then

(21) $$\frac{dy}{dx} = \sec^2 u \frac{du}{dx}.$$

The case of $y = \cot x$ is practically the same as $y = \tan x$. The graph of the function is shown in Fig. 10-7. To obtain the derived function of $y = \cot x$, we have only to note that

(22) $$y = \cot x = \frac{\cos x}{\sin x}$$

and apply the theorem on the derivative of a quotient of two functions. Again we leave the details for an exercise and note the result. If $y = \cot x$, then

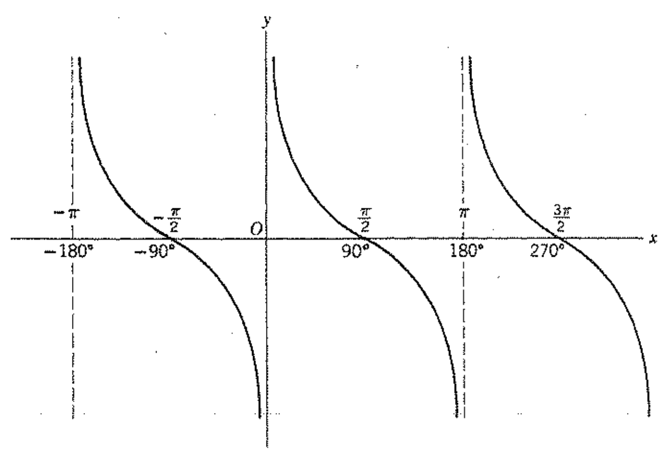
(23) $$\frac{dy}{dx} = -\csc^2 x,$$

and if $y = \cot u$, where u is a function of x, then

(24) $$\frac{dy}{dx} = -\csc^2 u \frac{du}{dx}.$$

The derived function in (23) fails to exist when x is any multiple of π (see Fig. 10-9).

Figure 10-7



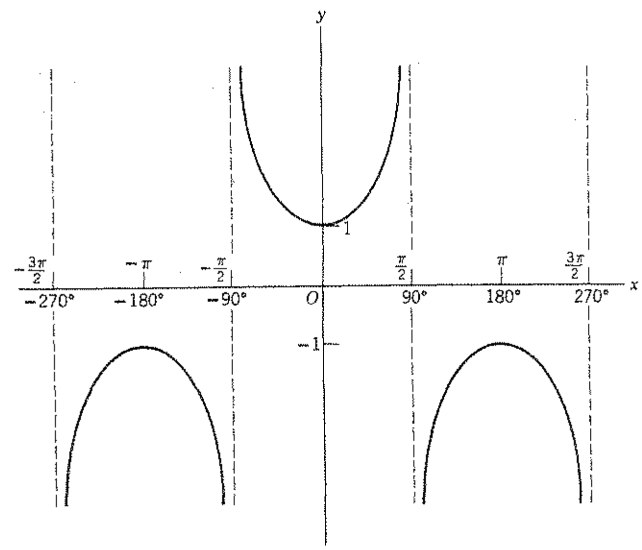


Figure 10-8

The derived function of $y = \sec x$, whose behavior is shown in Fig. 10-8, is also obtainable at once. Because

(25) $$y = \sec x = \frac{1}{\cos x},$$

the theorem on the derived function of a quotient of two functions can be applied and yields

(26) $$\frac{dy}{dx} = \sec x \tan x.$$

Also, if $y = \sec u$, where u is a function of x, then

(27) $$\frac{dy}{dx} = \sec u \tan u \frac{du}{dx}.$$

The derived function in (26) does not exist when x is an odd multiple of $\pi/2$.

Finally, we consider the sixth trigonometric function $y = \csc x$, whose behavior is shown in Fig. 10-9. The derived function is obtainable from the fact that

(28) $$y = \csc x = \frac{1}{\sin x}.$$

The result is

(29) $$\frac{dy}{dx} = -\csc x \cot x.$$

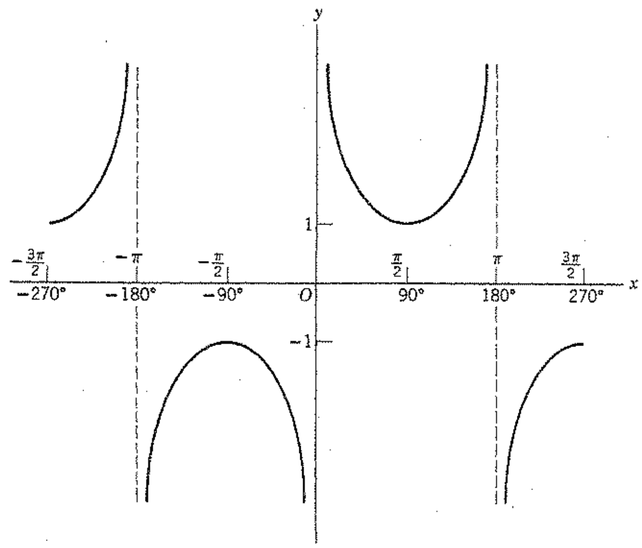


Figure 10-9

If $y = \csc u$, where u is a function of x, then

$$\frac{dy}{dx} = -\csc u \cot u \frac{du}{dx}. \tag{30}$$

The derived function in (29) fails to exist when x is any multiple of π.

Let us put together our results on the six functions. In all cases to be stated u is a function of x.

Theorem:

(31) If $y = \sin u$, then $\frac{dy}{dx} = \cos u \frac{du}{dx}$.

(32) If $y = \cos u$, then $\frac{dy}{dx} = -\sin u \frac{du}{dx}$.

(33) If $y = \tan u$, then $\frac{dy}{dx} = \sec^2 u \frac{du}{dx}$.

(34) If $y = \cot u$, then $\frac{dy}{dx} = -\csc^2 u \frac{du}{dx}$.

(35) If $y = \sec u$, then $\frac{dy}{dx} = \sec u \tan u \frac{du}{dx}$.

(36) If $y = \csc u$, then $\frac{dy}{dx} = -\csc u \cot u \frac{du}{dx}$.

EXERCISES

1. Find the derived functions of each of the following functions:

(a) $y = \sin 2x$. *Ans.* $dy/dx = 2\cos 2x$.

(b) $y = \cos 5x$.

(c) $y = 3\cos 2x$. *Ans.* $dy/dx = -6\sin 2x$.

(d) $y = 6\tan 5x$.

(e) $y = \sec 4x$. *Ans.* $dy/dx = 4\sec 4x \tan 4x$.

(f) $y = \sin x \cos x$.

(g) $y = \dfrac{1}{\cos 2x}$. *Ans.* $dy/dx = \dfrac{-2\sin 2x}{\cos^2 2x}$.

(h) $f(x) = \sin x \cot x$.

(i) $f(x) = \cos x \tan x$. *Ans.* $f'(x) = \cos x$.

(j) $y = \sqrt{1 - \cos^2 x}$

(k) $y = \sqrt{1 + \tan^2 x}$. *Ans.* $dy/dx = \sec x \tan x$.

(l) $y = \sqrt{2 - \cos^2 x}$. *Ans.* $dy/dx = \dfrac{\sin x \cos x}{\sqrt{1 + \sin^2 x}}$.

2. Find the derived functions of each of the following functions:

(a) $y = \sin^3 x$. *Ans.* $dy/dx = 3\sin^2 x \cos x$.

(b) $y = \sin^3 x \cos x$.

(c) $y = \sin^3 2x$. *Ans.* $dy/dx = 6\sin^2 2x \cos 2x$.

(d) $y = \sqrt[3]{\sin x}$.

(e) $y = \cos^2 2x$. *Ans.* $dy/dx = -4\sin 2x \cos 2x$.

(f) $y = \sin x^3$.

(g) $y = \tan 2x \cot 2x$. *Ans.* $y' = 0$.

(h) $y = \sin \dfrac{1}{x}$.

(i) $f(x) = x\cos\dfrac{1}{x}$. *Ans.* $f'(x) = \cos\dfrac{1}{x} + \dfrac{1}{x}\sin\dfrac{1}{x}$.

(j) $f(x) = \sqrt{\sin x^3}$.

(k) $y = \cos(\sin x)$. *Ans.* $dy/dx = -\sin(\sin x)\cos x$.

(l) $y = \sin^2 x + \cos^2 x$.

(m) $f(x) = \dfrac{\sin 2x}{\tan x}$. *Ans.* $f'(x) = -4\sin x \cos x$.

3. Given that $\dfrac{\Delta y}{\Delta x} = \dfrac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x}$ use the identity

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

to find dy/dx.

4. To differentiate $y = \cos x$, we might use a trigonometric identity and write $y = \sin\left(\frac{\pi}{2} - x\right)$. Obtain dy/dx from this form of $\cos x$.

5. Using the fact that $\tan x = \sin x/\cos x$, find y' when $y = \tan x$.

6. Using the fact that $\cot x = \cos x/\sin x$, find y' when $y = \cot x$.

7. Using the fact that $\sec x = 1/\cos x$, find y' when $y = \sec x$.

8. Using the fact that $\csc x = 1/\sin x$, find y' when $y = \csc x$.

9. Given $y = \sin x$, find d^2y/dx^2. *Ans.* $d^2y/dx^2 = -\sin x$.

10. Given $y = \cos x$, find d^2y/dx^2.

11. Let us accept for the present the fact that the range of a projectile fired from a gun which is inclined at an angle A to the ground is given by the formula $R = (V^2/16)\sin A \cos A$, where V, the initial velocity of the projectile, is fixed. Find the value of A for which the range is maximum. *Ans.* $\pi/4$.

12. Find the value of $\lim_{\Delta x \to 0} \frac{\sin(2x + 2\Delta x) - \sin 2x}{\Delta x}$.

13. Find the value of $\lim_{x \to \pi/2} \frac{\sin x - 1}{x - \pi/2}$. *Ans.* 0.

14. A mass M is drawn up a straight incline of given height h by a mass m which is attached to the first mass by a string passing from it over a pulley at the top of the incline (Fig. 10-10) and which hangs vertically. Find the angle of the incline in order that the time of ascent be a minimum.

Suggestion: The net force acting on M is $32m - 32M \sin A$.

Ans. $\sin A = m/2M$.

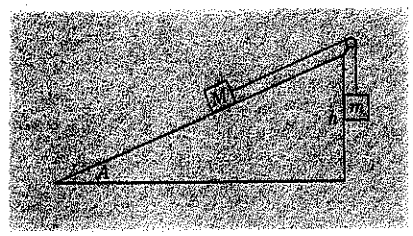


Figure 10-10

15. Given a point and a vertical line distant d from it, find the inclination of the straight line which would guide a particle acted on only by gravity from the point to the vertical line in the shortest time. *Ans.* $\pi/4$.

16. A swinging pendulum is 4 feet long and is rotating at the rate of 18°/sec when it makes an angle of 30° with the vertical. How fast is the end of the pendulum rising or falling at that moment? *Ans.* $\pi/5$ ft/sec.

17. An airplane, flying at an altitude of 2 miles, passes directly over an observer on the ground. A few moments later the observer notes that the airplane's angle of elevation is 30° and is decreasing at the rate of 15°/min. How fast is the plane traveling? *Ans.* $2\pi/3$ mi/min.

18. A revolving beacon 3600 feet off a straight shore makes 2 revolutions per minute. How fast does its beam sweep along the shore (a) at the point on the shore nearest the beacon? (b) at the point on the shore 4800 feet away from the beacon? *Ans.* (a) $14{,}400\pi$ ft/min.; (b) $25{,}600\pi$ ft/min.

19. A ferris wheel 50 feet in diameter makes 1 revolution every 2 minutes. If the center of the wheel is 30 feet above the ground, how fast is a passenger in the cab rising when he reaches a height of 40 feet?

Suggestion: Let A be the angle between the line from the center of the wheel to the cab and the line from the center of the wheel to the ground. Then the height of the cab above the ground is $h = 30 - 25 \cos A$.

Ans. $5\sqrt{21}\,\pi$ ft/min.

20. A destroyer at A sights a battleship at B, 2 miles away (Fig. 10-11). The latter is sailing due east at 10 mi/hr and the former is capable of sailing at 8 mi/hr. In what direction should the destroyer sail to come as close as possible to the battleship?

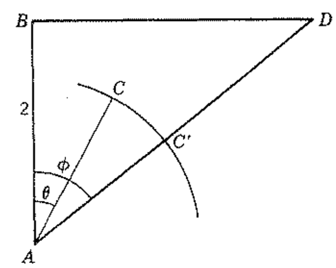


Figure 10-11

Suggestion: Suppose that the destroyer takes the direction θ and sails for some time t until it is closest to the battleship. The destroyer may then be at C and the battleship at D. The destroyer can do better by sailing in the direction ϕ, the angle determined by the straight line from A to D, for in that same time t it can travel the distance AC' equal to AC, and DC' is then less than DC. Hence it is necessary to consider only the situations in which the destroyer heads directly for the position of the battleship. However, this still leaves open a domain of possible values for ϕ, and the problem then becomes, which value of ϕ is best? *Ans.* $\sin \phi = 0.8$.

21. A steel girder 27 feet long is moved on rollers along a passageway and into a corridor 8 feet wide and at right angles to the passageway. How wide must the passageway be for the girder to go around the corner?

Suggestion: Idealize the girder as the line segment AB of Fig. 10-12. As the girder is moved around the corner it is best to keep it touching the inner vertex at 0 and touching the outer wall of the corridor. Then θ varies as the girder is moved into the corridor. The largest value of x as θ varies from 90° to 0° is the required width of the passageway. *Ans.* $5\sqrt{5}$.

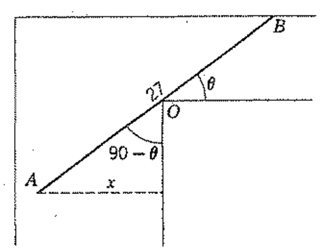


Figure 10-12

5. Integration of the Trigonometric Functions. Every new formula of differentiation gives us a new formula for integration. Thus from formulas (31) to (36) we have the following

Theorem: If $\dfrac{dy}{dx} = \sin u \dfrac{du}{dx}$, then

(37) $$y = \int \sin u \frac{du}{dx}\, dx = -\cos u + C.$$

If $\frac{dy}{dx} = \cos u \frac{du}{dx}$, then

(38) $$y = \int \cos u \frac{du}{dx}\, dx = \sin u + C.$$

If $\frac{dy}{dx} = \sec^2 u \frac{du}{dx}$, then

(39) $$y = \int \sec^2 u \frac{du}{dx}\, dx = \tan u + C.$$

If $\frac{dy}{dx} = \csc^2 u \frac{du}{dx}$, then

(40) $$y = \int \csc^2 u \frac{du}{dx}\, dx = -\cot u + C.$$

If $\frac{dy}{dx} = \sec u \tan \frac{du}{dx}$, then

(41) $$y = \int \sec u \tan u \frac{du}{dx}\, dx = \sec u + C.$$

If $\frac{dy}{dx} = \csc u \cot u \frac{du}{dx}$, then

(42) $$y = \int \csc u \cot u \frac{du}{dx} = -\csc u + C.$$

As in the case of the algebraic functions, integration is more difficult than differentiation. Let us consider a few examples involving the trigonometric functions.

Given

$$\frac{dy}{dx} = \cos 2x,$$

let us find y. If we think of $2x$ as u, we have $dy/dx = \cos u$. If we had in addition the factor du/dx, we could integrate by applying (38). In the present case, because du/dx is 2, we write

$$\frac{dy}{dx} = \tfrac{1}{2} \cos u \cdot 2.$$

Now, apart from the factor $\frac{1}{2}$, our derivative is in the form called for by (38). However, a constant factor can be kept separate and merely multiplied into the integral. Hence

$$y = \tfrac{1}{2} \sin u + C = \tfrac{1}{2} \sin 2x + C.$$

We may recall that the factor du/dx, which is 2 in this example, does not itself give rise to any term in the integral.