Our idealized formula in the insulation example does not take into account all possible variables. For instance, it does not reflect installation costs or the fact that some heat is lost through the walls and floor. Nevertheless, the problem as stated and solved is representative of fairly common conditions.

There is a general procedure we have used in solving the applied problems in this section. Below we list the major features of the procedure as a guide for you in solving applied problems involving extreme values.

1. After reading the problem carefully, choose a letter for the quantity to be maximized or minimized, and choose auxiliary variables for the other quantities appearing in the problem.
2. Express the quantity to be maximized or minimized in terms of the auxiliary variables. A diagram is often useful.
3. Choose one variable, say x, to serve as master variable, and use the information given in the problem to express all other auxiliary variables in terms of x. Again a diagram may be helpful.
4. Use the results of steps (2) and (3) to express the given quantity to be maximized or minimized in terms of x alone.
5. Use the theory of this chapter to find the desired maximum or minimum value. This usually involves finding a derivative and determining where it is 0, and then either evaluating the given quantity at endpoints and critical points or using Theorem 4.11 or 4.12.

EXERCISES 4.5

1. Find the two positive numbers whose sum is 18 and whose product is as large as possible.
2. Find the two real numbers whose difference is 16 and whose product is as small as possible.
3. A crate open at the top has vertical sides, a square bottom, and a volume of 4 cubic meters. If the crate has the least possible surface area, find its dimensions.
4. Suppose the crate in Exercise 3 has a top. Find the dimensions of the crate with minimum surface area.
5. Show that the entire region enclosed by the outdoor track in Example 3 has maximum area if the track is circular.
6. A Norman window is a window in the shape of a rectangle with a semicircle attached at the top (Figure 4.41). Find the dimensions that allow the maximum amount of light to enter, under the condition that the perimeter of the window is 12 feet.
7. Suppose a window has the shape of a rectangle with an equilateral triangle attached at the top. Find the dimensions that allow the maximum amount of light to enter, provided that the perimeter of the window is 12 feet.
8. A rectangle is inscribed in a semicircle of radius r, with one side lying on the diameter of the semicircle. Find the maximum possible area of the rectangle.

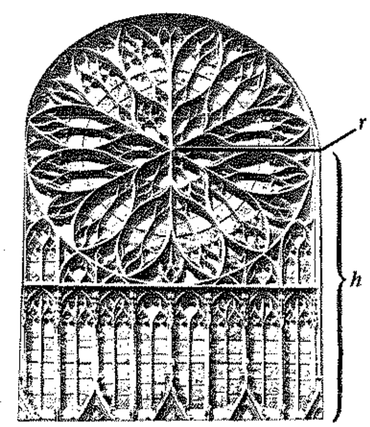


FIGURE 4.41

9. At 3 P.M. an oil tanker traveling west in the ocean at 15 kilometers per hour passes the same point as a luxury liner which arrived at the same spot at 2 P.M. while traveling north at 25 kilometers per hour. At what time were the ships closest together?
10. The coughing problem in Example 2 can be approached from a slightly different point of view. If v denotes the

velocity of the air in the windpipe, then F, r, and v are related by the equation

$$F = v(\pi r^2)$$

Consequently by (3),

$$v = \frac{F}{\pi r^2} = \frac{k}{\pi}(r_0 - r)r^2$$

Show that the velocity v is maximized when $r = \frac{2}{3}r_0$. (This shows that constriction of the windpipe during a cough appears to increase the air velocity in the windpipe and facilitate the cough.)

11. Find the point on the line $y = 2x - 4$ that is closest to the point $(1, 3)$.

12. Find the points on the parabola $y = x^2 + 2x$ that are closest to the point $(-1, 0)$.

13. Of all the triangles that pass through the point $(1, 1)$ and have two sides lying on the coordinate axes, one has the smallest area. Determine the lengths of its sides.

14. A horse breeder plans to set aside a rectangular region of 1 square kilometer for horses and wishes to build a wooden fence to enclose the region. Since one side of the region will run along a well-traveled highway, the breeder decides to make that side more attractive, using wood that costs three times as much per meter as the wood for the other sides. What dimensions will minimize the cost of the fence?

15. Suppose a landowner wishes to use 3 miles of fencing to enclose an isosceles triangular region of as large an area as possible. What should be the lengths of the sides of the triangle?

16. A manufacturer wishes to produce rectangular containers with square bottoms and tops, each container having a capacity of 250 cubic inches. If the material used for the top and the bottom costs twice as much per square inch as the material for the sides, what dimensions will minimize the cost?

17. A wire of length L is cut into two pieces. One piece is bent to form a square, and the other is bent to form a circle. Determine the minimum possible value for the sum A of the areas of the square and the circle. If the wire is actually cut, is there a maximum value of A?

18. A 12-foot wire is cut into 12 pieces, which are soldered together to form a rectangular frame whose base is twice as long as it is wide (as in Figure 4.42). The frame is then covered with paper.
    a. How should the wire be cut if the volume of the frame is to be maximized?
    b. How should the wire be cut if the total surface area of the frame is to be maximized?

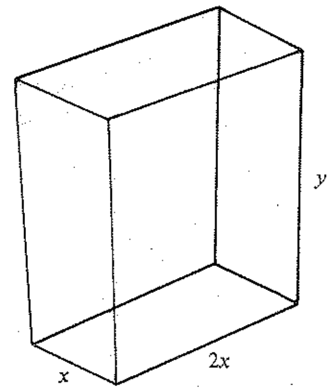


FIGURE 4.42

19. A company plans to invest \$50,000 for the next four years, and initially it buys oil stocks. If it seems profitable to do so, the oil stocks will be sold before the four-year period has lapsed, and the revenue from the sale of the stocks will be placed in tax-free municipal bonds. According to the company's analysis, if the oil stocks are sold after t years, then the net profit $P(t)$ in dollars for the four-year investment is given by

$$P(t) = 2(20 - t)^3 t \quad \text{for } 0 \le t \le 4$$

Determine whether the company should switch from oil stocks to municipal bonds, and if so, after what period of time.

20. Most post offices in the United States have the following limit on the size of a parcel that can be mailed by parcel post: The sum of the length of its longest side and its girth (the largest perimeter of a cross-section perpendicular to the longest side) can be no more than 84 inches.
    a. Find the dimensions of the rectangular parallelepiped with a square base having the largest volume that can be mailed. (There are two cases to be considered, depending on which side is longest.)
    b. Find the dimensions of the right circular cylinder having the largest volume that can be mailed. (Again, there are two cases to consider.)
    c. Find the dimensions of the cube having the largest volume that can be mailed.
    d. Show that it is possible for a parcel to be mailable and yet have a larger volume than a parcel that is not mailable. (*Hint:* Examine your solutions to (a)–(c).)
    e. For many small, rural, and army post offices, as well as post offices in Hawaii, Alaska, and Puerto Rico, the 84-inch limit is replaced by a 100-inch limit. Show that with the 100-inch limit it is still possible for a parcel to be mailable and yet have a larger volume than one that is not mailable.

21. If $C(x)$ is the cost of manufacturing an amount x of a given product and p is the price per unit amount, then the profit $P(x)$ obtained by selling an amount x is

$$P(x) = px - C(x)$$

(Notice that there is a loss if $P(x)$ is negative.)
   a. If $C(x) = cx$ and $c < p$, is there a maximum profit?
   b. If $C(x) = (x - 1)^2 + 2$, find the maximum profit.

22. Toward what point on the road should the ranger in Example 5 walk in order to minimize the travel time to the car if the car is located
   a. 10 miles down the road?
   b. $\frac{1}{2}$ mile down the road?
   c. an arbitrary number c of miles down the road?

23. It is known that homing pigeons fly faster over land than over water. Assume that they fly 10 meters per second over land but only 9 meters per second over water.
   a. If a pigeon is located at the edge of a straight river 500 meters wide and must fly to its nest, located 1300 meters away on the opposite side of the river (Figure 4.43), what path would minimize its flying time?
   b. If the nest were located 200 meters farther down the river, what path would minimize the flying time?

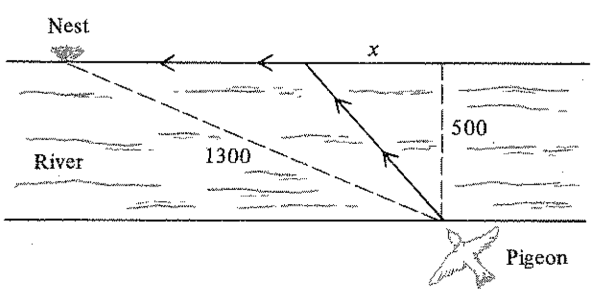


FIGURE 4.43

24. In an autocatalytic chemical reaction a substance A is converted into a substance B in such a manner that

$$\frac{dx}{dt} = kx(a - x)$$

where x is the concentration of substance B at time t, a is the initial concentration of substance A, and k is a positive constant. Determine the value of x at which the rate dx/dt of the reaction is maximum.

25. If we neglect air resistance, then the range of a ball (or any projectile) shot at an angle θ with respect to the x axis and with an initial velocity v_0 is given by

$$R(\theta) = \left(\frac{v_0^2}{g}\right)\sin 2\theta \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$$

where g is the acceleration due to gravity (32 feet per second per second).
   a. Show that the maximum range is attained when $\theta = \pi/4$.
   b. If $v_0 = 96$ feet per second and the aim is to snuff out a smouldering cigarette lying on the ground 144 feet away, at what angle should the ball be hit?
   c. The maximum height reached by the ball is

$$y_{\max} = \frac{(v_0^2 \sin^2\theta)}{2g}$$

   Why would it be a bad idea to hit the ball so that $y_{\max}$ is maximized?

26. A ring of radius a carries a uniform electric charge Q. The electric field intensity at any point x along the axis of the ring is given by

$$E(x) = \frac{Qx}{(x^2 + a^2)^{3/2}}$$

Find the maximum value of E.

27. If electric charge is uniformly distributed throughout a circular cylinder (such as a telephone wire) of radius a, then at any point whose distance from the axis of the cylinder is r, the electric field intensity is given by

$$E(r) = \begin{cases} cr & \text{for } 0 \le r \le a \\ ca^2/r & \text{for } r > a \end{cases}$$

where c is a positive constant.
   a. Show that $E(r)$ is maximum for $r = a$.
   b. Is E differentiable at a? Explain your answer.

28. An isosceles triangle has base 6 and height 12. Find the maximum possible area of a rectangle that can be placed inside the triangle with one side on the base of the triangle.

29. An isosceles triangle is inscribed in a circle of radius r. Find the maximum possible area of the triangle.

30. A cylindrical can with top and bottom has volume V. Find the radius of the can with the smallest possible surface area.

31. A cylinder is inscribed in a sphere with radius R. Find the height of the cylinder with the maximum possible volume.

32. A cylinder is inscribed in a cone of height H and base radius R. Determine the largest possible volume for the cylinder.

33. Find the radius of the cone with given volume V and minimum surface area. (*Hint:* The surface area S of a cone with radius r and height h is given by $S = \pi r\sqrt{r^2 + h^2}$.)

34. Three sides of a trapezoid are of equal length L, and no two are parallel. Find the length of the fourth side that gives the trapezoid maximum area.

35. A rectangular printed page is to have margins 2 inches wide at the top and the bottom and margins 1 inch wide on each of the two sides. If the page is to have 35 square inches of printing, determine the minimum possible area of the page itself.

36. A real estate firm can borrow money at 5% interest per year and can lend the money to its customers. If the amount of money it can lend is inversely proportional to the square of the interest rate at which it lends, what interest rate would maximize the firm's profit per year? (*Hint:* Let x be the loan interest rate. Notice that the profit is the product of the amount borrowed by the firm and the difference between the interest rates at which it lends and borrows.)

37. A company has a daily fixed cost of \$5000. If the company produces x units daily, then the daily cost in dollars for labor and materials is $3x$. The daily cost of equipment maintenance is $x^2/2{,}500{,}000$. What daily production minimizes the total daily cost per unit of production? (*Hint:* The cost per unit is the total cost $C(x)$ divided by x.)

38. A company sells 1000 units of a certain product annually, with no seasonal fluctuations in demand. It always reorders the same number x of units, stocks unsold units until no more remain, and then reorders again. If it costs b dollars to stock one unit for one year and there is a fixed cost of c dollars each time the company reorders, how many units should be reordered each time to minimize the total annual cost of reordering and stocking? (*Hint:* The company will have an average inventory of $x/2$ units and must reorder $1000/x$ times per year. Find the annual stocking and reordering costs and minimize their sum.)

39. Suppose we wish to estimate the probability p of rolling a 3 with a loaded die. We roll the die n times and obtain m 3's in a particular order. The probability of this is known to be $p^m(1-p)^{n-m}$. The ***maximum likelihood estimate*** of p based on the n rolls is the value of x that maximizes $x^m(1-x)^{n-m}$ on $[0, 1]$. Show that the maximum likelihood estimate of p is m/n.

°40. A farmer wishes to employ tomato pickers to harvest 62,500 tomatoes. Each picker can harvest 625 tomatoes per hour and is paid \$6 per hour. In addition, the farmer must pay a supervisor \$10 per hour and pay the union \$10 for each picker employed.
    a. How many pickers should the farmer employ to minimize the cost of harvesting the tomatoes?
    b. What is the minimum cost to the farmer?

41. Find the length of the largest thin, rigid pipe that can be carried from one 10-foot-wide corridor to a similar corridor at right angles to the first. Assume that the pipe has negligible diameter. (*Hint:* Find the length of the shortest line that touches the inside corner of the hallways and extends to the two walls.)

42. After work a person wishes to sit in a long park bounded by two parallel highways 300 meters apart. Suppose one highway is 8 times as noisy as the other. In order to have the quietest repose, how far from the quieter highway should the person sit? (*Hint:* The intensity of noise where the person sits is directly proportional to the intensity of noise at the source and inversely proportional to the square of the distance from the source.)

In Exercise 43 we present a mathematical problem that arises in two completely different settings (see Exercises 44 and 45).

43. Let p, q, and r be positive constants with $q < r$, and let

$$f(\theta) = p - q\cot\theta + \frac{r}{\sin\theta} \quad \text{for } 0 < \theta < \frac{\pi}{2}$$

Show that f has a minimum value on $(0, \pi/2)$ at the value of θ for which $\cos\theta = q/r$.

44. This problem derives from the biological study of vascular branching. Assume that a major blood vessel A leads away from the heart (P in Figure 4.44) and that in order for the heart to feed an organ at R, we must place an auxiliary artery somewhere between P and Q. The resistance $\mathcal{R}$ of the blood as it flows along the path PSR is given by

$$\mathcal{R}(\theta) = k\left[\frac{(a - b\cot\theta)}{r_1^4}\right] + k\left(\frac{b}{r_2^4\sin\theta}\right) \quad \text{for } 0 < \theta < \frac{\pi}{2}$$

where k, a, b, r_1, and r_2 are positive constants with $r_1 > r_2$ (see Figure 4.44). Where should the contact at S be made to produce the least resistance? (*Hint:* Using the result of Exercise 43, find the cosine of the angle θ for which $R(\theta)$ is minimized.)

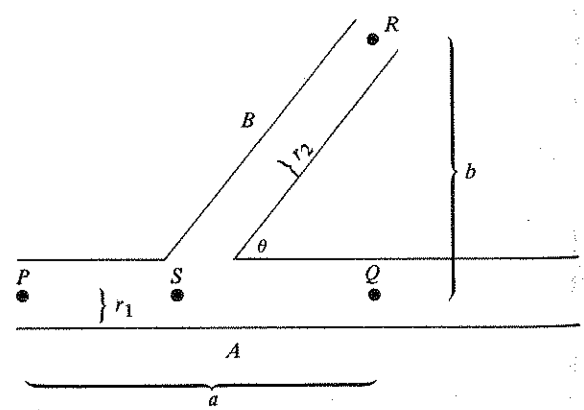


FIGURE 4.44

45. A bee's cell in a hive is a regular hexagonal prism open at the front, with a trihedral apex at the back (Figure 4.45). It can be shown that the surface area of a cell with apex θ is given by

$$S(\theta) = 6ab + \frac{3}{2}b^2\left(-\cot\theta + \frac{\sqrt{3}}{\sin\theta}\right) \quad \text{for } 0 < \theta < \frac{\pi}{2}$$

where a and b are positive constants. Show that the surface area is minimized if $\cos\theta = 1/\sqrt{3}$, so that $\theta \approx 54.7°$. (*Hint:* Use the result of Exercise 43.) Experiments have shown that bee cells have an average angle within 2′ (less than one tenth of one degree) of 54.7°.

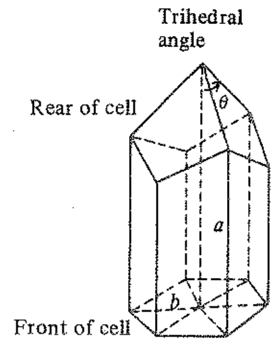
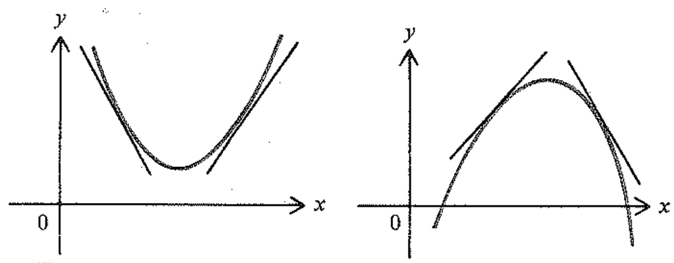


FIGURE 4.45

4.6 CONCAVITY AND INFLECTION POINTS

We now consider other ways the second derivative can help in graphing functions—through the notions of concavity and inflection points.



Tangent lines below graph

(a)

Tangent lines above graph

(b)

FIGURE 4.46

Concavity In Figure 4.46(a) the tangent lines lie below the graphs, whereas in Figure 4.46(b) the tangent lines lie above the graphs. To distinguish between these two cases, we define the notion of concavity.

DEFINITION 4.13

a. Let f be differentiable at c, and let l_c be the line tangent to the graph of f at $(c, f(c))$. The graph of f is ***concave upward*** at $(c, f(c))$ if there is an open interval I_c about c such that if x is in I_c and $x \neq c$, then $(x, f(x))$ lies above l_c. The graph of f is ***concave downward*** at $(c, f(c))$ if there is an open interval I_c about c such that if x is in I_c and $x \neq c$, then $(x, f(x))$ lies below l_c.