Math 181 Honors Practice Exam 1 Version A

~ Key ~

1. Find the domain of $f(x) = \sqrt{x^2+1}$.

$x^2+1 \geq 0$

is always true so domain is $(-\infty, \infty)$

2. Evaluate the sum $\sum_{k=2}^{6} \frac{k}{5}$. $= \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{5}{5} + \frac{6}{5}$

$= \frac{20}{5} = 4$

3. Compute in any way $\lim_{x \to 1} \frac{x}{x+3}$. $= \frac{1}{1+3} = \frac{1}{4}$

4. Compute in any way $\lim_{x \to \infty} \frac{2x+17}{x^2}$. $= \lim_{x \to \infty} \frac{2}{x} + \lim_{x \to \infty} \frac{17}{x^2}$

$= 0 + 0 = 0$

5. Convert the repeating decimal $1.\overline{43}$ to a fraction.

$$1.\overline{43} = 1 + \frac{43}{99} = \frac{142}{99}$$

6. Use induction to show $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for every positive integer n.

For $n=1$ then $1 = 1^2$ so the statement is true.

Suppose $1+3+5+\cdots+(2n-1)=n^2$

then

$$1+3+5+\cdots+(2n-1)+(2(n+1)-1)$$

$$= n^2 + 2n+1 = (n+1)^2$$

which completes the induction.

7. Solve the inequality $|2x-1|<5$.

$$-5<2x-1<5$$
$$-4<2x<6$$
$$-2<x<3$$

so $x \in (-2,3)$

8. Use δ-ϵ definition of limit to verify $\lim_{x\to 1}\frac{1}{x+2}=\frac{1}{3}$.

Let $\varepsilon>0$ and choose $\delta=\min(1,6\varepsilon)$

then $0<|x-1|<\delta$ implies

$$-1<x-1<1$$
$$+3 \quad\quad +3 \quad\quad +3$$
$$2<x+2<4 \quad \text{so} \quad \frac{1}{2}>\frac{1}{x+2}>\frac{1}{4}.$$

Therefore

$$\left|\frac{1}{x+2}-\frac{1}{3}\right|=\left|\frac{3-x-2}{3(x+2)}\right|=\frac{|x-1|}{3|x+2|}$$
$$<\frac{\delta}{3\cdot 2}\leq\varepsilon.$$

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9. Solve the inequality $\dfrac{x-5}{x-3} \le 2$.

$$\frac{x-5}{x-3} - 2 \le 0$$

$$\frac{x-5-2x+6}{x-3} \le 0$$

$$\frac{-x+1}{x+3} \le 0$$

$-x+1$	$+$	$-$	$-$
$x+3$	$-$	$-$	$+$
	$-$	$+$	$-$

Solution $x \in (-\infty, 1] \cup (3, \infty)$

10. Use the method of increments to find $\dfrac{dy}{dx}$ when $y = 2x^3$.

When x_0 changes to $x_0 + \Delta x$ we have

$$y_0 + \Delta y = 2(x_0 + \Delta x)^3$$

1
1 2 1
1 3 3 1

$$= 2\left(x_0^3 + 3x_0^2 \Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3\right)$$

$$= 2x_0^3 + 6x_0^2 \Delta x + 6x_0(\Delta x)^2 + 2(\Delta x)^3$$

Subtracting $y_0 = 2x_0^3$

$$\Delta y = 6x_0^2 \Delta x + 6x_0(\Delta x)^2 + 2(\Delta x)^3$$

Thus

$$\frac{\Delta y}{\Delta x} = 6x_0^2 + 6x_0 \Delta x + 2(\Delta x)^2 \to 6x_0^2$$

as $\Delta x \to 0$

Therefore $\dfrac{dy}{dx} = 6x^2$

11. Work only one of the following problems.

(i) Suppose $y = 1/u$ where u depends on the variable x. Use the method of increments to verify that

$$\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}.$$

(ii) Suppose

$$\lim_{x\to 1} f(x) = 2 \qquad \text{and} \qquad \lim_{x\to 1} g(x) = 3.$$

Use the δ-ϵ definition of limit to verify $\lim_{x\to 1}\big(f(x)g(x)\big) = 6$.

Answer to (i): When x_0 changes to $x_0+\Delta x$ then u_0 changes to $u_0+\Delta u$ and

$$y_0+\Delta y = \frac{1}{u_0+\Delta u}$$

$$(y_0+\Delta y)(u_0+\Delta u) = 1$$

$$y_0u_0 + \Delta y\, u_0 + \Delta u\, y_0 + \Delta y\,\Delta u = 1$$

Subtracting $y_0u_0 = 1$.

$$\Delta y\, u_0 + \Delta u\, y_0 + \Delta y\,\Delta u = 0$$

$$\Delta y(u_0+\Delta u) = -\Delta u\, y_0$$

$$\frac{\Delta y}{\Delta x} = \frac{-y_0}{u_0+\Delta u}\,\frac{\Delta u}{\Delta x}$$

Therefore taking $\Delta x \to 0$ so that $\Delta u \to 0$ we obtain

$$\frac{dy}{dx} = \frac{-y}{u}\frac{du}{dx} = \frac{-1}{u^2}\frac{du}{dx}.$$

Answer to (ii):

Let $\varepsilon > 0$.

Choose $\varepsilon_2 = \frac{1}{8}\varepsilon$.

Since $\lim_{x\to 1} f(x) = 2$ there is $\delta_2 > 0$ such that $0 < |x-1| < \delta_2$ implies $|f(x) - 2| < \varepsilon_2$

Choose $\varepsilon_3 = \min(1, \frac{1}{4}\varepsilon)$.

Since $\lim_{x\to 1} g(x) = 3$ there is $\delta_3 > 0$ such that $0 < |x-1| < \delta_3$ implies $|g(x) - 3| < \varepsilon_3$

Choose $\delta = \min(\delta_1, \delta_2)$ then $0 < |x-1| < \delta$ implies

$$
\begin{aligned}
|f(x)g(x) - 6| &= |f(x)g(x) - 2g(x) + 2g(x) - 6| \\
&\leq |g(x)|\,|f(x) - 2| + 2|g(x) - 3| \\
&< (|g(x) - 3| + 3)\,\varepsilon_2 + 2\varepsilon_3 \\
&< (\varepsilon_3 + 3)\varepsilon_2 + 2\varepsilon_3 \leq 4\varepsilon_2 + 2\varepsilon_3 \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$