~ Key ~

Math 181 Honors Exam 2 Version B

1. Compute the following derivatives using any method.

(i) $\frac{d}{dx}(x\sqrt{1+x})$ $= \sqrt{1+x} + \frac{x}{2\sqrt{1+x}}$

$= \frac{2+3x}{2\sqrt{1+x}}$

(ii) $\frac{d}{dx}\arctan(2x)$ $= \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$

(iii) $\frac{d}{dx}\ln(7+\sin x)$ $= \frac{1}{7+\sin x} \cdot \cos x$

$= \frac{\cos x}{7+\sin x}$

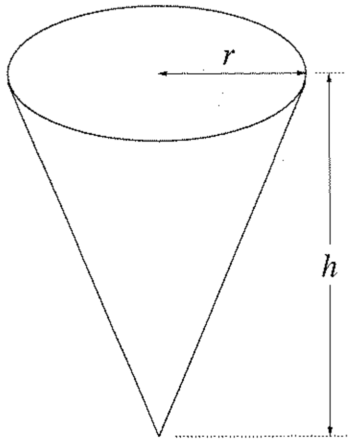
2. Use implicit differentiation to find $\frac{dy}{dx}$ where $y^2 + x^3 = 3 + |x+y|$.

$$2y\frac{dy}{dx} + 3x^2 = \frac{x+y}{|x+y|}\left(1+\frac{dy}{dx}\right)$$

$$\left(\frac{x+y}{|x+y|} - 2y\right)\frac{dy}{dx} = 3x^2 - \frac{x+y}{|x+y|}$$

$$\frac{dy}{dx} = \left(3x^2 - \frac{x+y}{|x+y|}\right) \Big/ \left(\frac{x+y}{|x+y|} - 2y\right)$$

3. Suppose fluid flows out of the bottom of a cone-shaped vessel at the rate of 3 ft^3/min. If the height of the cone is three times the radius, how fast is the height of the fluid decreasing when the fluid is 6 inches deep in the middle?



Convert inches to feet

$$V = \frac{1}{3} A \cdot h = \frac{\pi}{3} r^2 h = \pi r^3$$

$$h = 3r$$

$$\frac{dV}{dt} = 3$$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

$$\frac{dh}{dt} = 3 \frac{dr}{dt}$$

If $h = 6\text{in} = \frac{1}{2}\text{ft}$

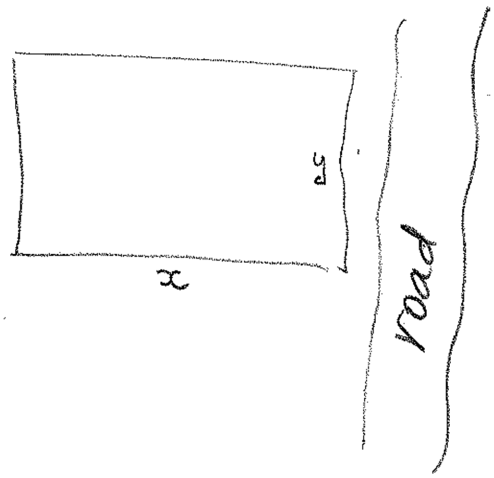
then $r = \frac{1}{6}\text{ft}$

so $$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi r^2} \frac{dV}{dt} = \frac{1}{\pi\left(\frac{1}{36}\right)} \cdot 3 = \frac{108}{\pi}$$

The height is decreasing at a rate of $\frac{108}{\pi}$ ft/min

4. A horse breeder plans to set aside a rectangular region of 1 square kilometer for horses and wishes to build a wooden fence to enclose the region. Since one side of the region will run along a well-traveled highway, the breeder decides to make that side more attractive, using wood that costs three times as much per meter as the wood for the other sides. What dimensions will minimize the cost of the fence?



Price of cheap wood is P
expensive " is 3P

$$C = yP + y3P + 2xP$$

$$C = P(4y + 2x)$$

$$A = xy = 1$$

$$C = P\left(4\frac{1}{x} + 2x\right)$$

$$\frac{dc}{dx} = P\left(-\frac{4}{x^2} + 2\right) = \frac{2P}{x^2}(x^2 - 2)$$

$$= \frac{2P}{x^2}(x - \sqrt{2})(x + \sqrt{2}) = 0$$

Thus $x = \sqrt{2}$ is where the minimum is.

$$y = \frac{1}{\sqrt{2}}$$

The dimensions that minimize the cost are $\sqrt{2}$ km $\times$ $\frac{1}{\sqrt{2}}$ km where the shorter side should be parallel to the road.

5. Use the δ-ϵ definition of limit to verify that $\lim_{x\to 5} \frac{1}{x} = \frac{1}{5}$.

Let $\varepsilon > 0$ be arbitrary and choose $\delta = \min(1, 20\varepsilon) > 0$

Then $0 < |x-5| < \delta$ implies

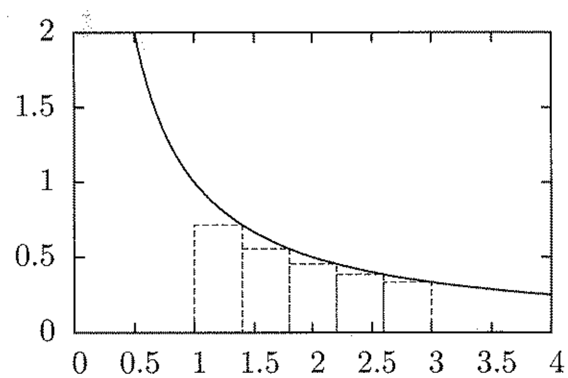
$-1 < x-5 < 1$ so $4 < x < 6$ so $\frac{1}{4} > \frac{1}{x} > \frac{1}{6}$.

Therefore

$$\left|\frac{1}{x} - \frac{1}{5}\right| = \left|\frac{5-x}{5x}\right| < \frac{\delta}{5|x|} < \frac{\delta}{20} \le \varepsilon,$$

Shows that $\lim_{x\to 5} \frac{1}{x} = \frac{1}{5}$.

6. Write the sum for the area of the five rectanges shown below that approximate $\ln 3$. Do not add up the terms or attempt to simplify the sum.



$w = (3-1)/5$. Therefore

$$A \approx \sum_{k=1}^{5} \frac{1}{1+k\frac{2}{5}} \cdot \frac{2}{5}$$

$$= \sum_{k=1}^{5} \frac{2}{5+2k}$$

$$= \frac{2}{7} + \frac{2}{9} + \frac{2}{11} + \frac{2}{13} + \frac{2}{15}.$$

7. Solve the following antidifferentiation problems:

(i) Find y so that $\frac{dy}{dx} = \cos(2x)$.

Let $u = 2x$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du} \cdot 2 = \cos u$

$\frac{dy}{du} = \frac{1}{2}\cos u$ implies $y = \frac{1}{2}\sin u + C$

In other words $y = \frac{1}{2}\sin(2x) + C$.

(ii) Find w so that $\frac{dw}{dt} = \frac{1}{t^4}$. $= t^{-4}$

Guess something like t^{-3}. Then

$\frac{dt^{-3}}{dt} = -3t^{-4}$ so $\frac{d}{dt}\left(\frac{-1}{3}t^{-3}\right) = t^{-4}$.

It follows that $w = -\frac{1}{3}t^{-3} + C$.

8. Compute the following limits in any way:

(i) $\lim_{x\to 1} \frac{1}{\sqrt{1+x}}$.

just plug in 1 since the function $\frac{1}{\sqrt{1+x}}$ is continuous at the point $x=1$.

$\lim_{x\to 1} \frac{1}{\sqrt{1+x}} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$.

(ii) $\lim_{x\to\infty} \frac{x-17}{1+x}$. $= \lim_{x\to\infty} \frac{1-\frac{17}{x}}{\frac{1}{x}+1} = \frac{1-0}{0+1} = 1$

9. [Extra Credit] Let f be a continuous function defined on the real line and define

$$y = \int_1^x f(t)\,dt.$$

Use the method of increments to show that $\dfrac{dy}{dx} = f(x)$.

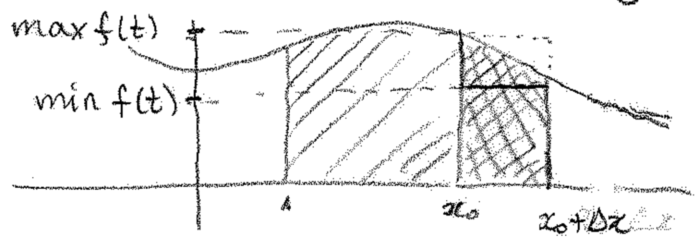
Consider the increment from x_0 to $x_0 + \Delta x$.
Then

$$y_0 + \Delta y = \int_1^{x_0+\Delta x} f(t)\,dt$$

and

$$y_0 = \int_1^{x_0} f(t)\,dt$$

The areas on the right hand sides of the above two equations are given graphically below



The difference is indicated by the ▩ marked piece.
Thus $\quad \Delta y = \displaystyle\int_{x_0}^{x_0+\Delta x} f(t)\,dt$

and therefore comparing the rectangular areas we have

$$(\Delta x)\left(\min_{t\in[x_0,x_0+\Delta x]} f(t)\right) \le \Delta y \le (\Delta x)\left(\max_{t\in[x_0,x_0+\Delta x]} f(t)\right).$$

Divide by Δx and take limits.

$$\min_{t\in[x_0,x_0+\Delta x]} f(t) \le \frac{\Delta y}{\Delta x} \le \max_{t\in[x_0,x_0+\Delta x]} f(t)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$f(x_0) \qquad\quad f(x_0) \qquad\quad f(x_0)$$

Therefore $\dfrac{\Delta y}{\Delta x} \to f(x_0)$ which implies $\dfrac{dy}{dx} = f(x)$.