Math 181 Honors Practice Exam 2 Version A

1. Compute the following derivatives using any method.

(i) $\frac{d}{dx}\left(\frac{\sin x}{2+\cos x}\right) = \frac{\left(\frac{d}{dx}\sin x\right)(2+\cos x) - \sin\left(\frac{d}{dx}(2+\cos x)\right)}{(2+\cos x)^2}$

$= \frac{(\cos x)(2+\cos x) + (\sin x)(\sin x)}{(2+\cos x)^2} = \frac{1+2\cos x}{(2+\cos x)^2}$

(ii) $\frac{d}{dx}\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)$

$y = \arcsin u$

$\sin y = u$

$(\cos y)\frac{dy}{du} = 1$

$\frac{dy}{du} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-u^2}}$

$u = \frac{1}{\sqrt{1+x^2}}$

$\frac{du}{dx} = -\frac{1}{2}(1+x^2)^{-3/2}\,2x$

$= \frac{-x}{(1+x^2)^{3/2}}$

$= \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{\sqrt{1-u^2}}\right)\left(\frac{-x}{(1+x^2)^{3/2}}\right)$

$= \frac{1}{\sqrt{1-\frac{1}{1+x^2}}}\cdot\frac{-x}{(1+x^2)^{3/2}}$

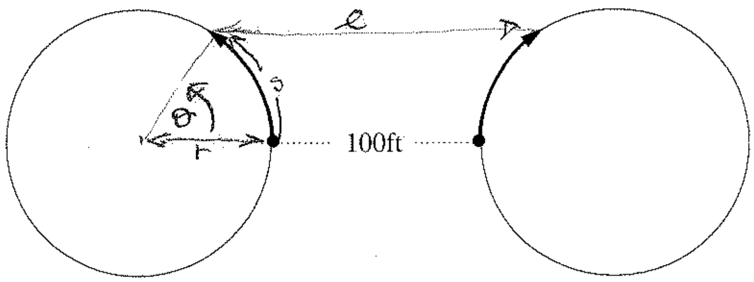
$= \frac{-x}{\sqrt{x^2}\cdot(1+x^2)} = \frac{-x}{|x|(1+x^2)}$

(iii) $\frac{d}{dx}\ln(1+2x) = \frac{1}{1+2x}\frac{d}{dx}(1+2x) = \frac{2}{1+2x}$

(iv) $\frac{d}{dx}(x^6+x^5+x^4+x^3+x^2+x+1)$

$= 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$

2. Two runners are running on circular tracks which have a circumference of 1320 feet and are 100 feet apart. The runners start at positions opposite and closest to each other as indicated. One runner runs clockwise at a constant rate of 880 feet/minute while the other runs counter clockwise at the same rate. How fast is the distance between the runners changing when each has run 165 feet?



$2\pi r = 1320$

$\theta r = s$

$\frac{d\theta}{dt} r = \frac{ds}{dt} = 880$

$\frac{d\theta}{dt} = \frac{880}{r}$

$\theta \frac{1320}{2\pi} = 165$

$\theta = \frac{330\pi}{1320}$

$= \frac{33\pi}{132} = \frac{\pi}{4}$

$\ell = 2r(1-\cos\theta) + 100$

$\frac{d\ell}{dt} = 2r\sin\theta \frac{d\theta}{dt}$

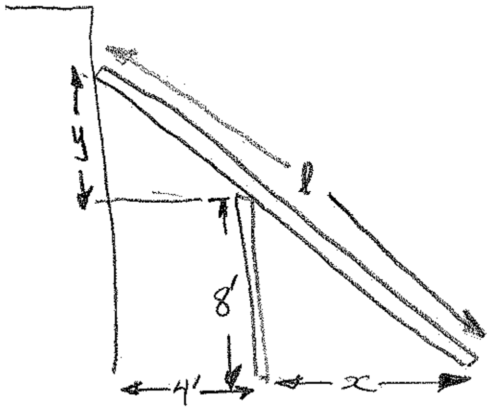
$= 2r\sin\theta \cdot \frac{880}{r}$

$= 2\sin\frac{\pi}{4} \cdot 880$

$= 2\frac{\sqrt{2}}{2} \cdot 880$

$= 880\sqrt{2}$

3. A fence 8 feet tall runs parallel to a tall building at a distance of 4 feet from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building.



Similar triangles: $\frac{y}{8} = \frac{4}{x}$

length of ladder:

$$\ell^2 = (4+x)^2 + (8+y)^2$$

$$= (4+x)^2 + 64\left(1+\frac{4}{x}\right)^2$$

Minimizing ℓ is the same as minimizing ℓ^2 and then taking the square root. Thus

$$\frac{d\ell^2}{dx} = 2(4+x) + 2\cdot 64\left(1+\frac{4}{x}\right)\left(\frac{-4}{x^2}\right) = 0$$

implies

$$x^3(4+x) - 256(x+4) = 0$$

so $\quad (4+x)(x^3-256) = 0$

Thus $x = -4$ or $x = \sqrt[3]{256} = 4\sqrt[3]{4}$

Since $x = -4 < 0$ makes no sense, then $x = 4\sqrt[3]{4}$ is where the minimum occurs.

$$\ell^2 = (4+4\sqrt[3]{4})^2 + 64\left(1+\frac{4}{4\sqrt[3]{4}}\right)^2$$

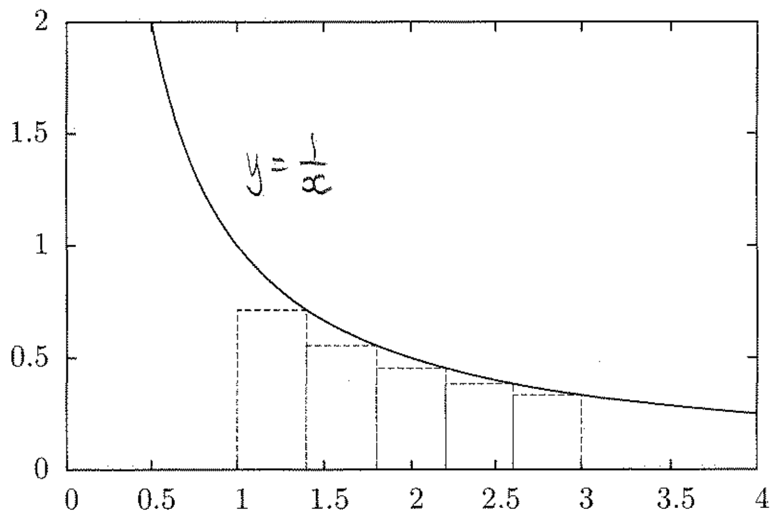
Thus the minimum ladder length is

$$\ell = \sqrt{(4+4\sqrt[3]{4})^2 + 64\left(1+\frac{4}{4\sqrt[3]{4}}\right)^2}$$

4. Convert the repeating decimal $1.\overline{36}$ to a fraction.

$$1.\overline{36} = 1 + \frac{36}{99} = \frac{99+36}{99} = \frac{135}{99}$$

5. Write the sum for the area of the five rectanges shown below that approximate $\ln 3$. Do not add up the terms or attempt to simplify the sum.



Width of rectangles $w = \frac{3-1}{5} = \frac{2}{5}$

$x_k = 1 + kw = 1 + k\frac{2}{5}$.

Area of 5 rectangles: $\displaystyle\sum_{n=1}^{5} \frac{1}{x_k} w = \sum_{n=1}^{5} \frac{1}{1+k\frac{2}{5}} \cdot \frac{2}{5}$

$$= \sum_{n=1}^{5} \frac{2}{5+2k} = \frac{2}{7} + \frac{2}{9} + \frac{2}{11} + \frac{2}{13} + \frac{2}{15}.$$

6. Solve the inequality $1 < \dfrac{2}{x} - \dfrac{2}{x+1}$.

$1-\frac{2}{x}+\frac{2}{x+1}<0$; $\quad \frac{x(x+1)-2(\cancel{x}+1)+\cancel{2x}}{x(x+1)}<0$; $\quad \frac{x^2+x-2}{x(x+1)}<0$

$\frac{(x-1)(x+2)}{x(x+1)}<0$

		-2		-1		0		1	
$x-1$	$-$		$-$		$-$		$-$		$+$
$x+2$	$-$		$+$		$+$		$+$		$+$
x	$-$		$-$		$-$		$+$		$+$
$x+1$	$-$		$-$		$+$		$+$		$+$
	$+$		$-$		$+$		$-$		$+$

Answer: $(-2,-1)\cup(0,1)$.

7. Use induction to show $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = n(n+1)(n+2)/3$.

FOR $n=1$: $\quad 1\cdot 2 = \frac{1\cdot 2\cdot 3}{3} = 2$ so it is true for $n=1$.

SUPPOSE for n that $1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$.
Claim that the result holds for $n+1$.

$$1\cdot 2 + 2\cdot 3 + \cdots + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

$$= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3}$$

which shows it is true for $n+1$.

This completes the induction.

8. Solve the following antidifferentiation problems:

(i) Find y so that $\dfrac{dy}{dx} = x^3 + 5$.

$y = \frac{1}{4}x^4 + 5x + C$

where C is an unknown constant.

(ii) Find w so that $\dfrac{dw}{dt} = \sin t$.

$w = -\cos t + C$

where C is an unknown constant.

9. Use the δ-ϵ definition of limit to verify that $\lim_{x\to 2} x^3 = 8$.

Let $\varepsilon > 0$ be arbitrary and choose $\delta = \min(1, \varepsilon/19)$

Then $0 < |x-2| < \delta$ implies.

$-1 < x < 1$

so

$1 < x < 3$.

Therefore

$|x^3 - 8| = |x^3 - 2x^2 + 2x^2 - 4x + 4x - 8|$

$\leq x^2|x-2| + |2x||x-2| + 4|x-2|$

$< (x^2 + |2x| + 4)\delta < (3^2 + 2\cdot 3 + 4)\delta$

$= 19\delta \leq \varepsilon$,

10. Use the method of increments to find $\frac{dy}{dx}$ when $y = \frac{1}{x}$.

Thus $xy = 1$ and

$$(x_0 + \Delta x)(y_0 + \Delta y) = 1$$

$$x_0 y_0 + (\Delta x) y_0 + x_0 \Delta y + (\Delta x)(\Delta y) = 1$$

subtract $x_0 y_0 = 1$

$$(\Delta x) y_0 + x_0 (\Delta y) + (\Delta x)(\Delta y) = 0$$

Therefore

$$(\Delta y)(x_0 + \Delta x) = -y_0 \Delta x$$

or $\frac{\Delta y}{\Delta x} = \frac{-y_0}{x_0 + \Delta x} \to \frac{-y_0}{x_0}$ as $\Delta x \to 0$

It follows $\frac{dy}{dx} = \frac{-y}{x} = -\frac{1}{x^2}$.

11. Use implicit differentiation to find $\frac{dy}{dx}$ where $y^3 + x^2 = \cos(xy)$.

$$3y^2 \frac{dy}{dx} + 2x = -\sin(xy) \frac{d}{dx}(xy)$$

$$3y^2 \frac{dy}{dx} + 2x = -\sin(xy)\left(y + x\frac{dy}{dx}\right)$$

Therefore

$$\left(3y^2 + x\sin(xy)\right)\frac{dy}{dx} = -2x - y\sin(xy)$$

so

$$\frac{dy}{dx} = -\frac{2x + y\sin(xy)}{3y^2 + x\sin(xy)}$$

12. Compute the following limits in any way:

(i) $\lim_{x\to\infty} \frac{x-17}{1+x^2}$. $= \lim_{x\to\infty} \frac{x-17}{1+x^2} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$

$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{17}{x^2}}{\frac{1}{x^2}+1} = \frac{0-0}{0+1} = \frac{0}{1} = 0$

(ii) $\lim_{x\to 0} \frac{1-\cos 3x}{x^2}$. $= \lim_{x\to 0} \frac{(1-\cos 3x)(1+\cos 3x)}{x^2(1+\cos 3x)}$

$= \lim_{x\to 0} \frac{1-\cos^2 3x}{x^2(1+\cos 3x)} = \lim_{x\to 0} \frac{\sin^2 3x}{x^2(1+\cos 3x)}$

$= 9 \cdot \lim_{x\to 0} \frac{\sin^2 3x}{(3x)^2} \cdot \lim_{x\to 0} \frac{1}{1+\cos 3x} = 9 \cdot \lim_{z\to 0} \frac{\sin z}{z} \lim_{z\to 0} \frac{\sin z}{z} \cdot \frac{1}{2} = \frac{9}{2}.$

13. Show that

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$$

using the identity $\sec^2 x = 1 + \tan^2 x$ and the fact that $\frac{d \tan x}{dx} = \sec^2 x$.

Let $y = \arctan x$ then $\tan y = x$

$$\frac{d}{dx}(\tan y) = \sec^2 y \frac{dy}{dx} = 1$$

Thus $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

In other words

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}.$$