Math 181 Final review answers

#1. Convert $7.\overline{5}$ into a fraction:

$$7+\frac{5}{9}=\frac{63+5}{9}=\frac{68}{9}$$

#2. Solve the equation $\ln(e^{x+1})=3$:

$$\ln(e^{x+1})=x+1=3, \qquad x=2$$

#3. Compute $\int_0^{\pi/2}\cos(3x)\,dx$

$$u=3x \qquad du=3dx$$

Thus

$$\int\cos(3x)dx=\frac{1}{3}\int\cos u\,du=\frac{1}{3}\sin u$$

$$=\frac{1}{3}\sin 3x$$

Therefore

$$\int_0^{\pi/2}\cos(3x)\,dx=\frac{1}{3}\sin 3x\Big|_0^{\pi/2}$$

$$=\frac{1}{3}\sin\frac{3\pi}{2}-\frac{1}{3}\sin 0=-\frac{1}{3}$$

#4 Use induction to show that

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

First, if $n=1$ then

$$\sum_{k=1}^{1}k=1 \quad\text{and}\quad \frac{1(1+1)}{2}=\frac{2}{2}=1$$

so the equation holds for $n=1$.

Now suppose, for induction, that the result holds for n. We need to prove it holds for $n+1$.

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

This completes the induction.

#5. Compute the sum $\sum_{k=5}^{29} (k+1)$

$$\sum_{k=5}^{29} (k+1) = \sum_{l=6}^{30} l = \sum_{l=1}^{30} l - \sum_{l=1}^{5} l = \frac{30(31)}{2} - \frac{5(6)}{2}.$$

where $l = k+1$.

#6. State the δ-ε definition of $\lim_{x\to p} f(x) = A$.

For every $\varepsilon > 0$ there is $\delta > 0$ such that $0 < |x-p| < \delta$ implies $|f(x) - A| < \varepsilon$.

#8. Compute the following limits.

(i) $\lim_{x\to 2} (x^2 - 7) = 4 - 7 = -3$

(ii) $$\lim_{x\to 4} \frac{\sin(x-4)}{\sqrt{x}-2} = \lim_{x\to 4} (\sqrt{x}+2)\cdot\frac{\sin(x-4)}{x-4}$$

$$= \left(\lim_{x\to 4} \sqrt{x}+2\right)\left(\lim_{h\to 0} \frac{\sin h}{h}\right) = 4\cdot 1 = 4$$

where $h = x - 4$.

(iii) $$\lim_{h\to 0}\left(\frac{1}{h^2+4h} - \frac{1}{4h}\right) = \lim_{h\to 0} \frac{4 - 4(h+1)}{4h(h+1)} = \lim_{h\to 0} \frac{-4h}{4h(h+1)} = \frac{-1}{1} = -1.$$

#7. Show $f(x) = 1/x$ is continuous at $p=1$ using the δ-ε definition of continuity.

This is the same as proving that

$$\lim_{x \to 1} \frac{1}{x} = 1$$

using the δ-ε definition of limit.

Let $\varepsilon > 0$ be arbitrary and choose $\delta = \min(\frac{1}{2}, \frac{\varepsilon}{2})$. Then $0 < |x-1| < \delta$ implies

$$-\frac{1}{2} < x-1 < \frac{1}{2} \quad \text{so} \quad \frac{1}{2} < x < \frac{3}{2} \quad \text{so} \quad 2 > \frac{1}{x} > \frac{2}{3}.$$

Therefore

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right| < \frac{1}{|x|}\delta < 2\delta \le \varepsilon.$$

#9. Explain the method of increments

The method of increments is a way of computing the derivative of a function by considering an increment from x_0 to $x_0 + \Delta x$ which results in a change from y_0 to $y_0 + \Delta y$. The rate of change $\Delta y / \Delta x$ is computed and then the limit is found when $\Delta x \to 0$. This give the instantaneous rate of change.

#10 Suppose $y = uv$ where u and v depend on x. Show that

$$\frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$$

using the method of increments.

Let x change from x_0 to $x_0 + \Delta x$.
Then u changes from u_0 to $u_0 + \Delta u$,
and v — " — v_0 to $v_0 + \Delta v$.

It follows that

$$y_0 + \Delta y = (u_0 + \Delta u)(v_0 + \Delta v) = u_0 v_0 + \Delta u\, v_0 + u_0 \Delta v + \Delta u\, \Delta v$$

and $\quad y_0 = u_0 v_0$.

subtracting yields

$$\Delta y = \Delta u\, v_0 + u_0 \Delta v + \Delta u\, \Delta v$$

Therefore

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} v_0 + u_0 \frac{\Delta v}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

as $\Delta x \to 0$ then $\frac{\Delta u}{\Delta x} \to \frac{du}{dx}$ and $\frac{\Delta v}{\Delta x} \to \frac{dv}{dx}$. Also $\Delta u \to 0$. It follows that

$$\frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx} + 0 \cdot \frac{dv}{dx}$$

Thus $\quad \frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$.

#11 Compute the following derivatives:

(i) $\frac{d}{dx}(x^5 + 3x^4 - x^2 + 4) = 5x^4 + 12x^3 - 2x$

(ii) $\frac{d}{dx} \sin(\ln x) = \cos(\ln x) \cdot \frac{1}{x}$

(iii) $\frac{d}{dx} \int_1^{1+x^2} \frac{\sin t}{t}\, dt = \left(\frac{d}{du} \int_1^u \frac{\sin t}{t}\, dt\right)\left(\frac{du}{dt}\right)$

by the chain rule where $u = 1 + x^2$. Now by the fundamental theorem of Calculus we have

$$\frac{d}{du} \int_1^u \frac{\sin t}{t}\, dt = \frac{\sin u}{u} = \frac{\sin(1+x^2)}{1+x^2}$$

It follows that $\quad \frac{d}{dx} \int_1^{1+x^2} \frac{\sin t}{t}\, dt = \frac{\sin(1+x^2)}{1+x^2} \cdot 2x$.

#12 Using the method of increments prove the Fundamental Theorem of Calculus.

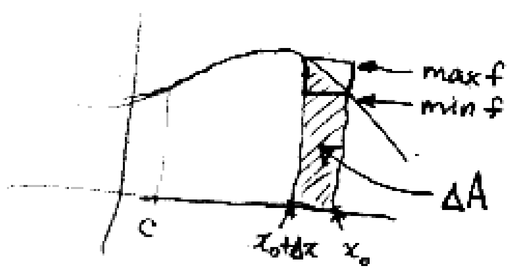
<u>Theorem:</u> Let f be a function that is integrable on $[a,x]$ for each $x \in [a,b]$. Let c be such that $a \le c \le b$ and define a new function A as

$$A = \int_c^x f(t)\,dt \qquad \text{if } a \le x \le b.$$

Then the derivative dA/dx exists at each point in the open interval (a,b) where f is continuous and for such x we have that $dA/dx = f(x)$.

Let x change from x_0 to $x_0 + \Delta x_0$. Then A changes from A_0 to $A_0 + \Delta A$ as

$$A_0 + \Delta A = \int_c^{x_0+\Delta x} f(t)\,dt$$



subtract $\quad A_0 = \int_c^{x_0} f(t)\,dt$

$$\Delta A = \int_{x_0}^{x_0+\Delta x} f(t)\,dt.$$

Now comparing ΔA to the area of the rectangles

$$\left(\min_{x\in[x_0,x_0+\Delta x]} f(x)\right)\Delta x \le \Delta A \le \left(\max_{x\in[x_0,x_0+\Delta x]} f(x)\right)\Delta x$$

Thus as $\Delta x \to 0$ we have

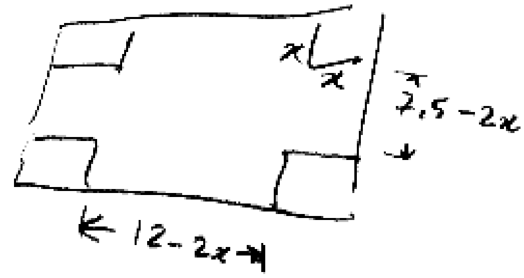
$$\min_{x\in[x_0,x_0+\Delta x]} f(x) \le \frac{\Delta A}{\Delta x} \le \max_{x\in[x_0,x_0+\Delta x]} f(x)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \quad \text{as } \Delta x \to 0$$

$$f(x_0) \qquad\qquad f(x_0) \qquad\qquad f(x_0)$$

Therefore $dA/dx = f(x)$.

#13 An open box is made from a rectangular piece of material by removing equal squares at each corner and turning up the sides. Find the dimensions of the box of largest volume that can be made in this manner if the material has sides 7.5 and 12 inches.



$$V = x(7.5-2x)(12-2x)$$
$$= x(15-4x)(6-x)$$
$$= x(4x-15)(x-6) = x(4x^2-39x+90) = 4x^3-39x^2+90x$$

$15 + 24 = 39 \qquad 15 \times 6 = 90$

$$\frac{dV}{dx} = 12x^2-78x+90 = 6(2x^2-13x+15) = 6(2x-3)(x-5) = 0$$

so $x = 3/2$ or $x = 5$.

Since $x=5$ would result in a negative length and negative volume $x = 3/2$ is the maximum.

The dimensions of the box are $3/2 \times 4.5 \times 6$.

#14. Convert $1.\overline{27}$ into a fraction.

$$1 + \frac{27}{99} = \frac{99+27}{99} = \frac{126}{99}$$

#15. Use the δ-ε definition of limit to verify $\lim_{x\to 4} \sqrt{x} = 2$.

Let $\varepsilon > 0$ be arbitrary and choose $\delta = \min(1, (\sqrt{3}+2)\varepsilon)$.
Then $0 < |x-4| < \delta$ implies

$$-1 < x-4 < 1 \text{ so } 3 < x < 5 \text{ so } \sqrt{3} < \sqrt{x} < \sqrt{5} \text{ so } \sqrt{3}+2 < \sqrt{x}+2 < \sqrt{5}+2$$

$$\text{so } \frac{1}{\sqrt{3}+2} > \frac{1}{\sqrt{x}+2} > \frac{1}{\sqrt{5}+2}$$

Therefore

$$|\sqrt{x}-2| = \frac{|x-4|}{\sqrt{x}+2} < \frac{\delta}{\sqrt{3}+2} \le \varepsilon.$$

#17. Use the method of increments to find dy/dx where $y = 1/x$.

$$y_0 + \Delta y = \frac{1}{x_0 + \Delta x} \quad , \quad y_0 = \frac{1}{x_0}$$

so $$(y_0 + \Delta y)(x_0 + \Delta x) = 1$$

Therefore
$$y_0 x_0 + \Delta y x_0 + y_0 \Delta x + \Delta y \Delta x = 1$$
$$x_0 y_0 = 1$$

Subtract

$$\Delta y x_0 + y_0 \Delta x + \Delta y \Delta x = 0$$

$$\Delta y x_0 = -(y_0 + \Delta y)\Delta x$$

$$\frac{\Delta y}{\Delta x} = \frac{-y_0 + \Delta y}{x_0} \to -\frac{y_0}{x_0} \quad \text{as } \Delta x \to 0$$

since $\Delta y \to 0$ as $\Delta x \to 0$. It follows that

$$\frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{x^2}.$$

#16. Suppose $\lim_{x \to 2} f(x) = 5$. Use the δ-ε definition of limit to verify that $\lim_{x \to 2} x f(x) = 10$.

Let $\varepsilon > 0$. Since $\lim_{x \to 2} f(x) = 5$ then for $\varepsilon_2 = \varepsilon/6 > 0$ there is $\delta_2 > 0$ such that $0 < |x-2| < \delta_2$ implies $|f(x) - 5| < \varepsilon_2$. Now choose $\delta = \min(1, \delta_2, \varepsilon/10)$. Then $0 < |x-2| < \delta$ implies
$-1 < x-2 < 1$ so $1 < x < 3$.

Therefore
$$|xf(x) - 10| \le |xf(x) - 5x + 5x - 10| \le |x||f(x) - 5| + 5|x-2|$$
$$\le 3\varepsilon_2 + 5\delta \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

#18. Use the limit $\lim_{h\to 0}\frac{\sin h}{h}=1$ to compute $\lim_{h\to 0}\frac{1-\cos 3h}{h^2}$.

$$\lim_{h\to 0}\frac{1-\cos 3h}{h^2}=\lim_{h\to 0}\frac{1-\cos^2 3h}{h^2(1+\cos 3h)}=\left(\lim_{h\to 0}\frac{\sin^2 3h}{h^2}\right)\left(\lim_{h\to 0}\frac{1}{1+\cos 3h}\right)$$

$$=9\left(\lim_{h\to 0}\frac{\sin^2 3h}{(3h)^2}\right)\cdot\frac{1}{2}=9\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)=9.$$

where $x=3h$

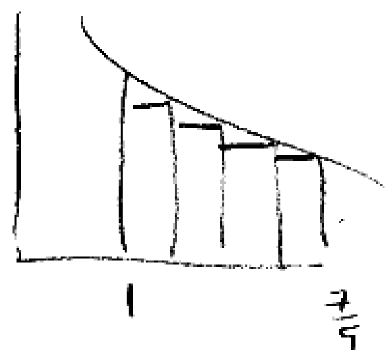
#19 Use the fundamental theorem of Calculus to compute $\frac{d}{dt}\int_0^t \sin(x^2)\,dx$

$$\frac{d}{dt}\int_0^t \sin(x^2)\,dx=\sin(t^2)$$

#20 Write the sum for the area of the four rectangles shown below that approximate the area under the curve $f(x)=1/x$ between $x=1$ and $x=7/5$. Do not add up the terms or simplify

$$\Delta x=\frac{7/5-1}{4}=\frac{2/5}{4}=\frac{1}{10}$$

$$A\approx\frac{1}{10}\left(\frac{1}{1.1}+\frac{1}{1.2}+\frac{1}{1.3}+\frac{1}{1.4}\right).$$



#21. Find the following derivatives.

(i) $\frac{d}{dx}\arctan(2\sqrt{x}+3)=\frac{1}{1+(2\sqrt{x}+3)^2}\frac{d}{dx}(2\sqrt{x})=\frac{1}{1+(2\sqrt{x}+3)^2}\cdot\frac{1}{\sqrt{x}}$

(ii) $\frac{d}{dx}\sin(x^2+3)=\cos(x^2+3)\cdot 2x$

(iii) $\frac{d}{dx}\tan\left(\frac{x}{x^4+7}\right)=\sec^2\left(\frac{x}{x^4+7}\right)\cdot\frac{d}{dx}\frac{x}{x^4+7}=\sec^2\left(\frac{x}{x^4+7}\right)\cdot\frac{x^4+7-x(4x^3)}{(x^4+7)^2}$

(iv) $\frac{d^4}{dx^4}(x^4+17x^3+27x^2+13x+7)=4\cdot3\cdot2=24$

#22. Find the following sums.

(i) $\sum_{k=3}^{9} k = 3+4+5+6+7+8+9 = 42$

(ii) $\sum_{k=n^2}^{n^3+1} k = \sum_{k=1}^{n^3+1} k - \sum_{k=1}^{n^2-1} k = \frac{(n^3+1)(n^3+2)}{2} - \frac{(n^2-1)n^2}{2}$

(iii) $\sum_{k=n}^{n+17} k^2 = \sum_{k=1}^{n+17} k^2 - \sum_{k=1}^{n-1} k^2 = \frac{(n+17)(n+18)(2(n+17)+1)}{6} - \frac{(n-1)n(2(n-1)+1)}{6}$

(iv) $\sum_{k=1}^{31} (k-16)^2 = \sum_{k=1}^{31}(k^2-32k+256) = \frac{(31)(32)(63)}{6} - 32\cdot\frac{(31)(32)}{2} + 256\cdot 31$.

23. Find the following definite and indefinite integrals.

(i) $\int (x^2-1)^2\,dx = \int (x^4-2x^2+1)\,dx = \frac{x^5}{5} - \frac{2}{3}x^3 + x + C$

(ii) $\int x\cos(2x^2+5)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(2x^2+5)+C.$

$u = 2x^2+5 \quad du = 4x\,dx$

(iii) $\int_0^1 5x^{99}\,dx = \frac{5}{100}x^{100}\Big|_0^1 = \frac{5}{100} = \frac{1}{20}$

(iv) $\int_0^2 x\sqrt{x+3}\,dx$

First find the antiderivative.

$\int x\sqrt{x+3}\,dx = \int (u+3)\sqrt{u}\,du = \int (u^{3/2} - 3u^{1/2})\,du = \frac{2}{5}u^{5/2} - 2u^{3/2} + C$

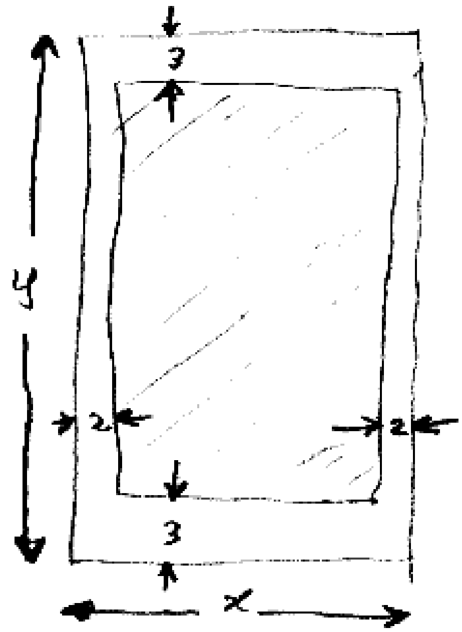
$u = x+3 \quad du = dx$

Now find the area using fundamental theorem

$\int_0^2 x\sqrt{x+3}\,dx = \frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2}\Big|_0^2$

$= \frac{2}{5}(5)^{5/2} - 2(5)^{3/2} - \frac{2}{5}(3)^{5/2} + 2(3)^{3/2}$

#24. A certain poster requires 96 in² for the printed message and must have 3-in margins at the top and bottom and a 2-in margin on each side. Find the overall dimensions of the poster if the amount of paper used is to be minimum.



$$\begin{cases} (x-4)(y-6) = 96 \\ A = xy \end{cases}$$

Solve for y using constraint:

$$y-6 = \frac{96}{x-4}, \quad y = 6 + \frac{96}{x-4}$$

Then

$$A = x\left(6 + \frac{96}{x-4}\right) = 6x + \frac{96x}{x-4}$$

$$\frac{dA}{dx} = 6 + \frac{96(x-4) - 96x}{(x-4)^2} = 6 - \frac{4 \cdot 96}{(x-4)^2} = 0$$

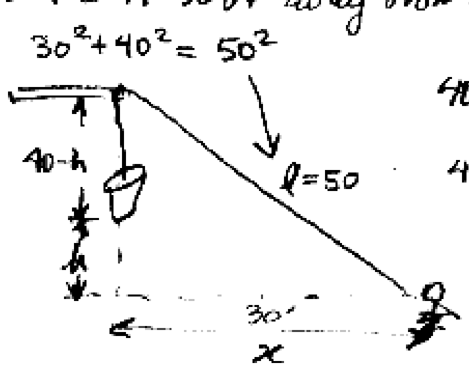
Then $(x-4)^2 = \frac{4 \cdot 96}{6} = 2 \cdot 32 = 64$

$x-4 = 8$ so $x = 12$

$y = 6 + \frac{96}{8} = 6 + 12 = 18$

The overall dimensions of the poster are 12x18 inches.

#25 A woman raises a bucket of a cement to a platform 40 ft above her head by means of a rope 80 ft long that passes over a pulley on the platform. If she holds her end of the rope at head level and walks away at 5 ft/sec, how fast is the bucket rising when she is 30 ft away from the spot directly below the pulley.



$40 - h = 80 - l$ so $\frac{dh}{dt} = \frac{dl}{dt}$

$40^2 + x^2 = l^2$ so $2x\frac{dx}{dt} = 2l\frac{dl}{dt}$

$$\frac{dh}{dt} = \frac{dl}{dt} = \frac{x}{l}\frac{dx}{dt} = \frac{30}{50} \cdot 5\text{ ft/sec} = 3\text{ ft/sec}.$$

#26. Find the following antiderivatives:

(i) $\int \frac{1}{x^2-9}\,dx = \int\left(\frac{A}{x-3} + \frac{B}{x+3}\right)dx = A\ln|x-3| + B\ln|x+3| + C$

where
$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$
$$1 = x(A+B) + 3(A-B)$$

Thus $A+B=0$ and $3(A-B)=1$

$B=-A$ so $3(A+A)=1$ so $A=\frac{1}{6}$ and $B=-\frac{1}{6}$

The final answer is
$$\int \frac{1}{x^2-9}\,dx = \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$

(ii) $\int \frac{1}{x^2+9}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx = \frac{1}{3}\int \frac{1}{1+u^2}\,du$

$u=\frac{x}{3}$ $\quad du = \frac{1}{3}dx$

$= \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan\frac{x}{3} + C.$

(iii) $\int xe^x\,dx$

Since $\frac{d}{dx}(xe^x) = xe^x + e^x$

$\left(\frac{d}{dx}(xe^x)\right) - e^x = xe^x$

$\frac{d}{dx}(xe^x - e^x) = xe^x$

Therefore $\int xe^x\,dx = xe^x - e^x + C.$

(iv) $\int x\cos(x^2+14)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2+14) + C.$

$u = x^2+14$ $\quad du = 2x\,dx$