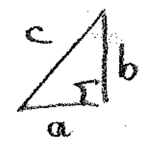
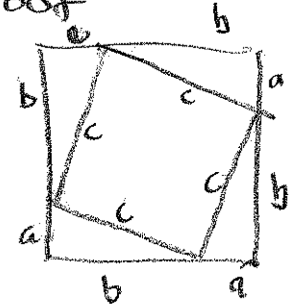
Math 181

1 Pythagorean theorem.

If 

then $a^2+b^2=c^2$.

Proof



$$(a+b)^2 = 4\left(\tfrac{1}{2}ab\right) + c^2$$

$$a^2+\cancel{2ab}+b^2 = \cancel{2ab}+c^2$$

$$a^2+b^2 = c^2$$

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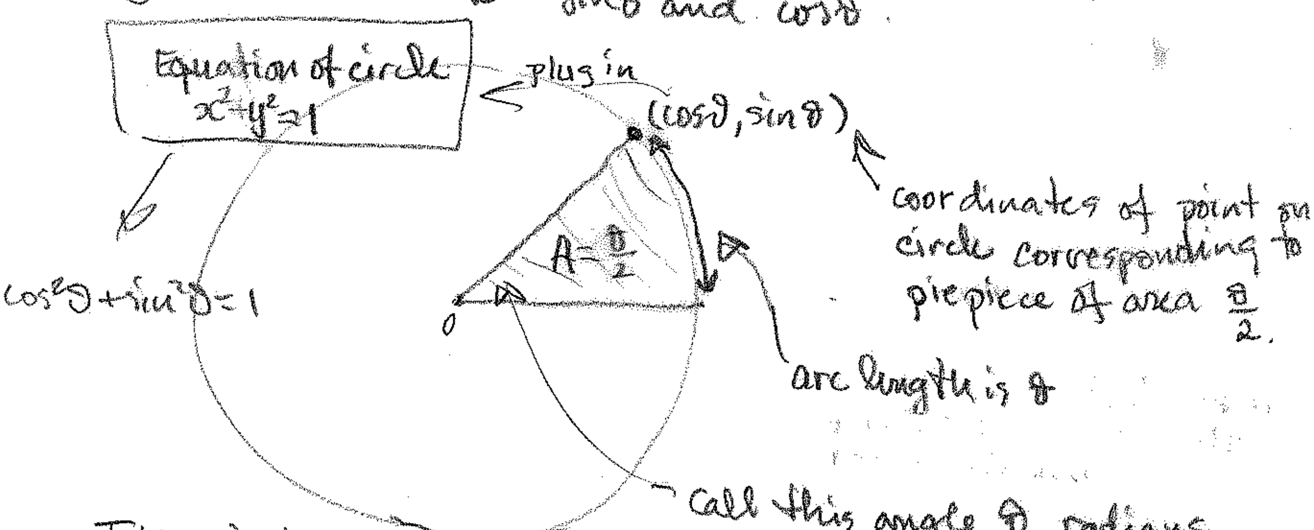
2 Definition of $\sin\theta$ and $\cos\theta$.



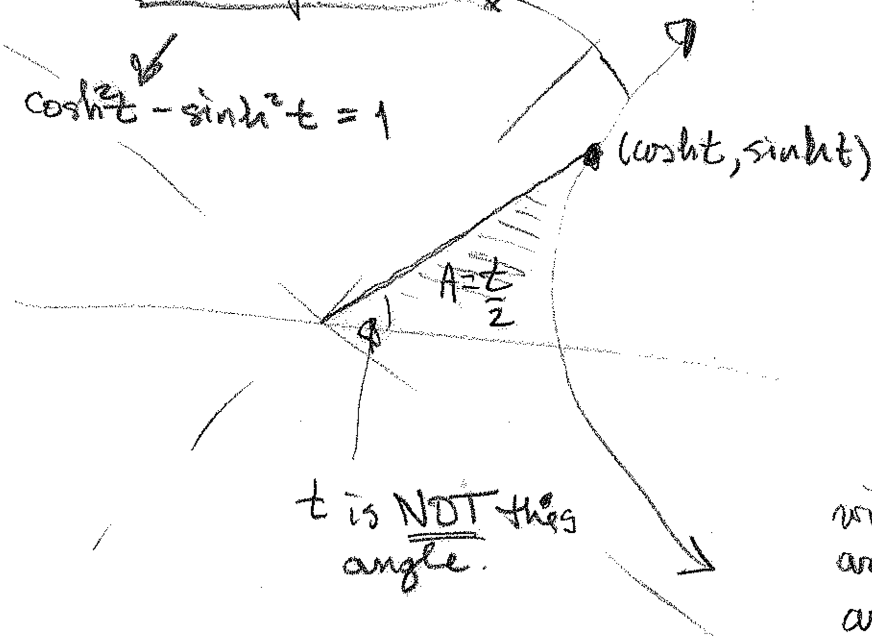
It's obvious the angle, arclength and area are proportional.

The fact that the arc length is θ when the area is $\frac{\theta}{2}$ is something you may remember but it is not obvious and a result of integral calculus which we'll derive later.

③ Definition of sinht and cosht.

$\boxed{\text{Eq of hyperbola: } x^2 - y^2 = 1}$

$\cosh^2 t - \sinh^2 t = 1$



coordinates of point on hyperbola corresponding to piece of area $\frac{t}{2}$

with a hyperbola the arc length, angle and area are NOT proportional.

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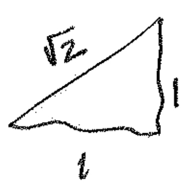
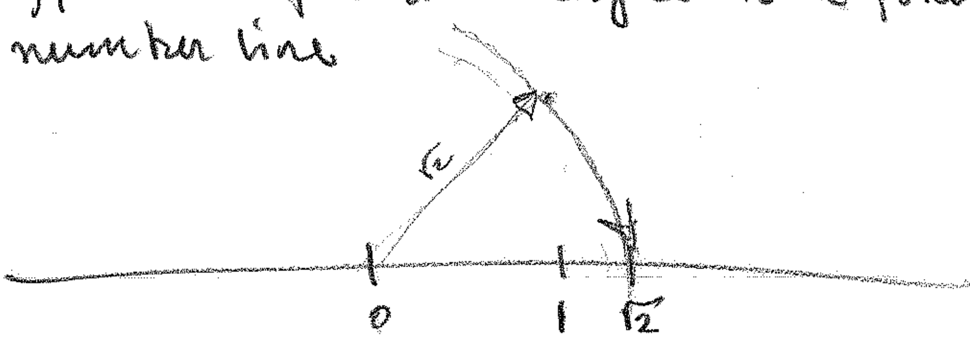
④



$a^2 + b^2 = c^2$

$1 + 1 = c^2$

The hypotenuse of this triangle is a point on the number line



5 Numbers.

Natural numbers: 1, 2, 3, 4, ...

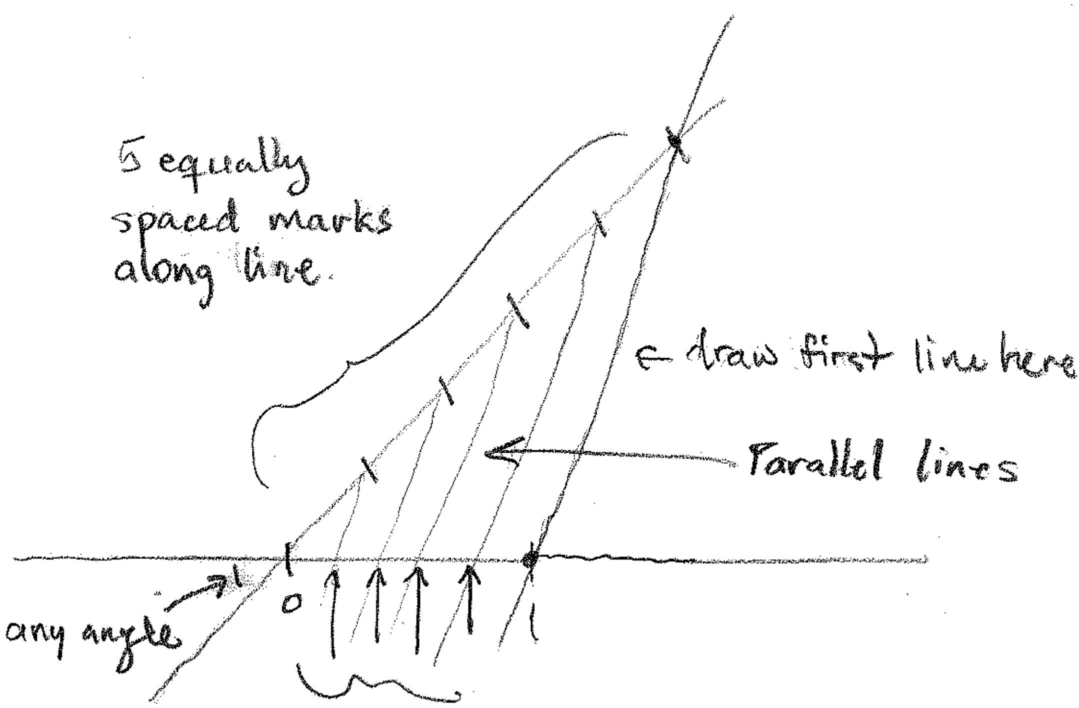
Integers: $0, \pm 1, \pm 2, \pm 3, \ldots$

Rational numbers $\frac{n}{m}$

where n, m are integers and $m \neq 0$.

Rational numbers are also known as fractions

6 Geometric construction to make $\frac{1}{5}$.



The resulting subdivisions of the line.

⑦ What is an irrational number?

The Greeks discovered that $\sqrt{2}$ can not be expressed as a fraction. This led Greek scientists to rely on geometry rather than arithmetic and algebra.

Proof that $\sqrt{2}$ is irrational.

see text pages 5–6.

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⑧ Correspondence between fractions and repeating decimals,

$$\begin{array}{r} .\overline{142857} \\ 7\,\big)\,1.0 \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 10 \end{array}$$

repeats

So $\frac{1}{7} = .\overline{142857}$

8 Decimal to fraction.

$$.\overline{5} = \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \cdots$$

$$= 5\Big(\underbrace{\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots}_{S}\Big)$$

$$S = \frac{1}{10} + \cancel{\frac{1}{10^2}} + \cancel{\frac{1}{10^3}} + \cancel{\cdots}$$

$$\frac{1}{10}S = \cancel{\frac{1}{10^2}} + \cancel{\frac{1}{10^3}} + \cancel{\frac{1}{10^4}} + \cancel{\cdots}$$

Subtract these equations

cancels when subtracted

$$\left(1-\frac{1}{10}\right)S = \frac{1}{10}$$

$$\frac{9}{10}S = \frac{1}{10}$$

$$S = \frac{1}{9}$$

Therefore

$$.\overline{5} = \frac{5}{9}$$

8 Another example.

$$.\overline{36} = \frac{36}{100} + \frac{36}{100^2} + \frac{36}{100^3} + \cdots$$

$$= 36\underbrace{\left(\frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \cdots\right)}_{S}$$

$$S = \frac{1}{100} + \cancel{\frac{1}{100^2}} + \cancel{\frac{1}{100^3}} + \cancel{\cdots}$$

$$\frac{1}{100}S = \cancel{\frac{1}{100^2}} + \cancel{\frac{1}{100^3}} + \cancel{\frac{1}{100^4}} + \cdots$$

subtract

cancel

$$\frac{99}{100}S = \frac{1}{100}$$

$$S = \frac{1}{99}$$

$$.\overline{36} = \frac{36}{99} = \frac{12}{33} = \frac{4}{11}$$

D) Completing the square.

Solve: $x^2 + 1 < x$

$$x^2 - x + 1 < 0$$

$$\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + 1 < 0$$

$$\underbrace{\left(x - \tfrac{1}{2}\right)^2}_{\text{positive}} + \underbrace{\tfrac{3}{4}}_{\text{positive}} < 0$$

There are no values of x such that $x^2+1<x$.

Solve: $x^2 - 8x > 1$

$$x^2 - 8x - 1 > 0$$

$$(x-4)^2 - 16 - 1 > 0$$

$$(x-4)^2 - 17 > 0$$

Factor as difference of squares

$$(x-4-\sqrt{17})(x-4+\sqrt{17}) > 0$$

		$4-\sqrt{17}$		$4+\sqrt{17}$	
$(x-4-\sqrt{17})$	$-$		$-$		$+$
$(x-4+\sqrt{17})$	$-$		$+$		$+$
P	$+$		$-$		$+$

Solution is $(-\infty, 4-\sqrt{17}) \cup (4+\sqrt{17}, \infty)$

10 Inequalities involving polynomials

<u>Solve:</u>

$$\underbrace{(x-1)(x-2)(x+4)(x-6)} > 0$$

zeros are $x = 1, 2, -4, 6$, plot on number line.

sign of each factor in each box

	-4		1		2		6	
factors:								
$(x-1)$	$-$		$-$		$+$		$+$	$+$
$(x-2)$	$-$		$-$		$-$		$+$	$+$
$(x+4)$	$-$		$+$		$+$		$+$	$+$
$(x-6)$	$-$		$-$		$-$		$-$	$+$
P	$+$		$-$		$+$		$-$	$+$

product of the signs go here

Pick positive since ">"

Solution is $(-\infty, -4) \cup (1, 2) \cup (6, \infty)$

(11) Geometric-arithmetic mean inequality

$$\sqrt{xy} \le \frac{x+y}{2}$$

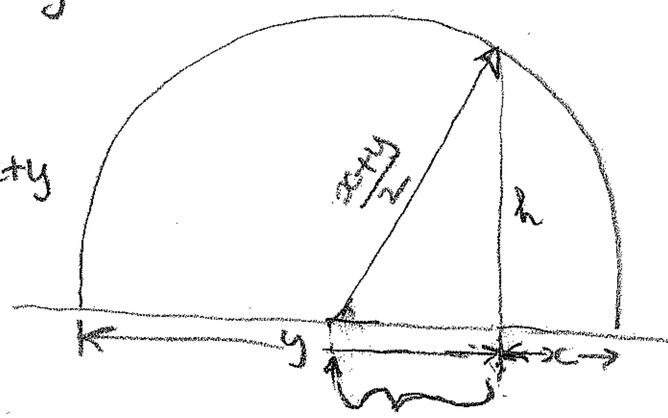
Algebraic proof

$$0 \le (a-b)^2 = a^2 - 2ab + b^2$$
$$2ab \le a^2 + b^2$$
$$ab \le \frac{a^2+b^2}{2}$$

Now let $a = \sqrt{x}$ and $b = \sqrt{y}$.

Geometric proof

diameter is $x+y$
radius is $\frac{x+y}{2}$



This distance
$\frac{x+y}{2} - x = \frac{y-x}{2}$

Pythagorean theorem

$$\left(\frac{y-x}{2}\right)^2 + h^2 = \left(\frac{x+y}{2}\right)^2$$

$$\frac{\cancel{y^2} - 2xy \cancel{+x^2}}{4} + h^2 = \frac{\cancel{x^2} + 2xy \cancel{+y^2}}{4}$$

Thus $h^2 = xy$ or $h = \sqrt{xy}$

Therefore $\sqrt{xy} \le \frac{x+y}{2}$.

(12) Monotone functions:

Monotone increasing

$x < y$ implies $f(x) < f(y)$

or

$f(x) < f(y)$ whenever $x < y$.

Monotone decreasing

$x < y$ implies $f(x) > f(y)$

non-decreasing

$x < y$ implies $f(x) \leq f(y)$

non-increasing

$x < y$ implies $f(x) \geq f(y)$.