# Rational root theorem

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In algebra, the **rational root theorem** (or 'rational root test' to find the zeros) states a constraint on solutions (or roots) to the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

with integer coefficients.

Let $a_0$ and $a_n$ be nonzero. Then each rational solution $x$, when written as a fraction $x = p/q$ in lowest terms, i.e. gcd(p,q) = 1, satisfies

- $p$ is an integer factor of the constant term $a_0$, and
- $q$ is an integer factor of the leading coefficient $a_n$.

Thus, a list of possible rational roots of the equation can be derived using the formula $x = \pm\frac{p}{q}$.

For example, every rational solution of the equation

$$3x^3 - 5x^2 + 5x - 2 = 0$$

must be among the numbers symbolically indicated by

$$\pm \tfrac{1,2}{1,3},$$

which gives the list of possible answers:

$$1, -1, 2, -2, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}.$$

These root candidates can be tested, for example using the Horner scheme. In this particular case there is exactly one rational root. If a root candidate does not satisfy the equation, it can be used to shorten the list of remaining candidates. E.g., x = 1 does not satisfy the equation as the left hand side equals 1. This means that substituting x = 1 + t yields a polynomial in t with constant term 1, while the coefficient of t^3 remains the same as the coefficient of x^3. Applying the Rational Roots theorem thus yields the following possible roots for t:

$$t = \pm\tfrac{1}{1,3}$$

Therefore,

$$x = 1 + t = 2, 0, \frac{4}{3}, \frac{2}{3}$$

Root candidates that do not occur on both lists are ruled out. The list of rational root candidates has thus shrunk to just x = 2 and x = 2/3.

If a root $r_1$ is found, the Horner scheme will also yield a polynomial of degree $n - 1$ whose roots, together with $r_1$, are exactly the roots of the original polynomial.

It may also be the case that none of the candidates is a solution; in this case the equation has no rational solution. The fundamental theorem of algebra states that any polynomial with integer (or real, or even complex) coefficients must have at least one root in the set of complex numbers. Any polynomial of odd degree (degree being $n$ in the example above) with real coefficients must have a root in the set of real numbers.

If the equation lacks a constant term $a_0$, then 0 is one of the rational roots of the equation.

The theorem is a special case (for a single linear factor) of Gauss's lemma on the factorization of polynomials.

The **integral root theorem** is a special case of the rational root theorem if the leading coefficient $a_n$=1.

# A Proof

Suppose p/q is a root to our integer coefficient polynomial, f(x), of degree n, where p and q are integers. Suppose p/q is fully reduced so that both p and q are coprime (greatest common divisor of p and q is 1 - gcd(p,q) = 1).

Then, f(p/q) = 0. Multiplying both sides of this by q^(n-1), yields

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q^(n-1) * f(p/q) = a_n * p^n / q + {sum of integers} = 0.
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Hence,

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a_n * p^n / q = {integer}.
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Because p and q are coprime, q divides a_n. Similarly, suppose we had multiplied by q^n/p instead. Then we have q^n * f(p/q) / p = {sum of integers} + a_o * q^n / p = 0. Rearranging, we get

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a_o * q^n / p = {sum of integers).
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Because p and q are coprime p divides a_o. Thus p is an integer factor of the constant term, q is an integer factor of the leading coefficient, and p/q is a root of f(x).

# External links