1 (i) Find the domain of $f(x) = \dfrac{1}{\sqrt{x^2 - 3x - 5}}$.

f is undefined when either the denominator is zero, or when the radicand is negative. Thus the domain of f is the set of x such that $x^2 - 3x - 5 > 0$. Solving this, we have

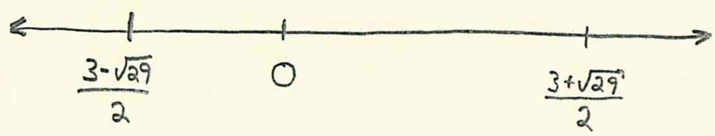
$$x^2 - 3x - 5 = 0$$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9 + 20}}{2}$$

$$= \frac{3 \pm \sqrt{29}}{2}.$$

Thus there are three intervals of interest:



If $x = 0$, $x^2 - 3x - 5 = -5 < 0$, so the interval containing 0 is not part of the domain. We can easily check that the other two intervals are in the domain. Then the domain of f is given by

$$\left(-\infty, \frac{3 - \sqrt{29}}{2}\right) \cup \left(\frac{3 + \sqrt{29}}{2}, \infty\right).$$

■

① (ii) Find the domain of $g(x) = \left(\frac{1}{2 + \sin(x)} \right)^2$.

g is undefined whenever the denominator is zero, that is, when $2 + \sin(x) = 0$. Note that $\sin(x) \in [-1, 1]$ for any $x \in \mathbb{R}$. This means that $2 + \sin(x) \in [1, 3]$, so $2 + \sin(x)$ is always greater that 0. Therefore the domain of g is $\mathbb{R}$. $\blacksquare$

① (iii) Find the domain of $h(x) = x/x$.

h is undefined whenever the denominator is zero. The denominator is 0 only when $x = 0$. Thus the domain is given by $\mathbb{R} \setminus \{0\}$.

$\blacksquare$

(2) For the next three problems, the following result is helpful:

If $|x|<1$ and n is a natural number, then

$$x^n + x^{n+1} + x^{n+2} + \cdots = \frac{x^n}{1-x} \qquad (*)$$

Can you verify this result?

(i)
$$3.4\overline{5} = 3 + \frac{4}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \cdots$$

$$= 3 + \frac{4}{10} + 5\left(\frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots\right)$$

$$= 3 + \frac{4}{10} + 5\left(\left(\tfrac{1}{10}\right)^2 + \left(\tfrac{1}{10}\right)^3 + \left(\tfrac{1}{10}\right)^4 + \cdots\right)$$

$$= 3 + \frac{4}{10} + 5\left(\frac{(1/10)^2}{1-1/10}\right) \qquad \text{by } (*)$$

$$= 3 + \frac{4}{10} + 5\left(\frac{1}{100}\cdot\frac{10}{9}\right)$$

$$= 3 + \frac{4}{10} + \frac{5}{90}$$

$$= \frac{270}{90} + \frac{36}{90} + \frac{5}{90}$$

$$= \boxed{\frac{311}{90}}$$

(ii)
$$0.0\overline{63} = \frac{63}{1000} + \frac{63}{100000} + \frac{63}{10000000} + \cdots$$

$$= \frac{63}{10}\left(\left(\tfrac{1}{100}\right) + \left(\tfrac{1}{100}\right)^2 + \left(\tfrac{1}{100}\right)^3 + \cdots\right)$$

$$= \frac{63}{10}\left(\frac{1/100}{1-1/100}\right) \qquad \text{by } (*)$$

$$= \frac{63}{10}\cdot\left(\frac{1}{100}\cdot\frac{100}{99}\right)$$

$$= \frac{63}{990} = \boxed{\frac{7}{110}}$$

(iii) $19.\overline{9} = 19 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

$$= 19 + 9\left(\left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots\right)$$

$$= 19 + 9\left(\frac{1/10}{1 - 1/10}\right) \qquad \text{by } (*)$$

$$= 19 + 9\left(\frac{1}{9}\right)$$

$$= 19 + 1 = \boxed{20}$$

(3) (i) $[1, 2, 3] = 1 + \cfrac{1}{2 + \cfrac{1}{3}}$

$$= 1 + \frac{1}{7/3}$$

$$= 1 + \frac{3}{7}$$

$$= \boxed{\frac{10}{7}}$$

(ii) $[1, \overline{1, 2}] = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$

Let $x = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2+\cdots}}}$

Note that

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{x}} .$$

Solving for x, we have

$$x = 1 + \cfrac{1}{\frac{2x}{x} + \frac{1}{x}}$$

$$= 1 + \frac{x}{2x+1}$$

$$x(2x+1) = (2x+1) + x$$

$$2x^2 + x = 3x + 1$$

$$2x^2 - 2x - 1 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4+8}}{2(2)} = \frac{1 \pm \sqrt{3}}{2}$$

$$[1, \overline{1, 2}] = 1 + \frac{1}{x}$$

$$= 1 + \frac{2}{1 \pm \sqrt{3}}$$

$$= \frac{1 \pm \sqrt{3} + 2}{1 \pm \sqrt{3}}$$

$$= \frac{3 \pm \sqrt{3}}{1 \pm \sqrt{3}} \left(\frac{1 \mp \sqrt{3}}{1 \mp \sqrt{3}} \right)$$

$$= \frac{3 - 3\sqrt{3} \mp \sqrt{3} - 3}{1 - 3}$$

$$= \frac{-3\sqrt{3} \pm \sqrt{3}}{-2}$$

$$= \frac{-2\sqrt{3}}{-2} \text{ or } \frac{-4\sqrt{3}}{-2}$$

$$= \sqrt{3} \text{ or } \tfrac{1}{2}\sqrt{3}$$

We know that $[1, \overline{1,2}] > 1$, and $\frac{1}{2}\sqrt{3} < 1$, thus it must be the case that

$$[1, \overline{1, 2}] = \sqrt{3} .$$

(iii) $[2, \overline{3}] = 2 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}$

Let $x = \cfrac{1}{3 + \cfrac{1}{3+\cdots}}$. Then $x = \dfrac{1}{3+x}$. Solving for x, we have

$$x = \frac{1}{3+x}$$

$$3x + x^2 = 1$$

$$x^2 + 3x - 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{9+4}}{2} = \frac{-3 \pm \sqrt{13}}{2}.$$

$$[2, \overline{3}] = 2 + x$$

$$= 2 + \frac{-3 \pm \sqrt{13}}{2}$$

$$= \frac{4 - 3 \pm \sqrt{13}}{2}$$

$$= \frac{1 \pm \sqrt{13}}{2}$$

$[2, \overline{3}] > 0$, so it must be that

$$[2, \overline{3}] = \frac{1 + \sqrt{13}}{2}$$

④ (i) Let $\varepsilon > 0$ and choose $\delta < \varepsilon/3$. Then for any $x \in \mathbb{R} \setminus \{2\}$ such that $|x-2| < \delta$, we have

$$|3x-6| = 3|x-2| < 3\delta < 3\frac{\varepsilon}{3} = \varepsilon.$$

Therefore $\lim_{x \to 2} 3x = 6.$ ∎

④ (ii) Let $\varepsilon > 0$ and choose $\delta < \min\{1, \varepsilon\}$. Then for any $x \in \mathbb{R} \setminus \{-2, 3\}$ such that $|x-3| < \delta$, the following holds:

① As $|x-3| < \delta < 1$, it follows that

$$-1 < x-3 < 1 \implies 20 < 5x+10 < 30$$
$$\implies \frac{1}{20} > \frac{1}{5x+10} > \frac{1}{30}$$

Thus $1/|5x+10| < 1.$

Then

$$\left|\frac{1}{2+x} - \frac{1}{5}\right| = \left|\frac{5-2-x}{10+5x}\right|$$
$$= \frac{|3-x|}{|5x+10|}$$
$$< |x-3| \quad \text{(by ① above)}$$
$$< \delta$$
$$< \varepsilon \quad \text{(by the selection of } \delta)$$

Therefore

$$\lim_{x \to 3} \frac{1}{2+x} = \frac{1}{5}.$$ ∎

4 (iii) Let $\varepsilon > 0$ and fix $\delta < \min\{1, \varepsilon\}$. Then for any $x \in [-4, \infty) \setminus \{5\}$ such that $|x-5| < \delta$, we know that

$$-1 < x - 5 \Rightarrow 8 < x + 4$$
$$\Rightarrow \sqrt{8} < \sqrt{x+4}$$
$$\Rightarrow 1 > \frac{1}{\sqrt{8}} > \frac{1}{\sqrt{x+4}+3}. \qquad (*)$$

Then

$$\begin{aligned}
|\sqrt{4+x} - 3| &= \left| \frac{4+x-9}{\sqrt{4+x}+3} \right| \\
&= \frac{|x-5|}{|\sqrt{4+x}+3|} \\
&< |x-5| && \text{by } (*) \\
&< \delta \\
&< \varepsilon && \text{by the choice of } \delta
\end{aligned}$$

Therefore

$$\lim_{x \to 5} \sqrt{4+x} = 3. \qquad \blacksquare$$

(5) Let $\varepsilon > 0$ and choose $\delta < \min\{1, \varepsilon/10\}$ so small that if $x \in \mathcal{D}(f)\setminus\{5\}$ and $|x-2| < \delta$, then $|f(x) - 5| < \varepsilon/6$. * Then, as $|x-2| <$ and $\delta < 1$, we have that $1 < x < 3$. It then follows that

$$
\begin{aligned}
|x f(x) - 10| &= |x f(x) - 5x + 5x - 10| \\
&= |x(f(x) - 5) + 5(x-2)| \\
&\le |x(f(x)-5)| + |5(x-2)| \quad \text{(triangle inequality)} \\
&= \underbrace{|x|}_{<3}\,\underbrace{|f(x) - 5|}_{<\varepsilon/6} + 5\underbrace{|x-2|}_{<\delta} \\
&< 3\frac{\varepsilon}{6} + 5\delta \\
&< 3\frac{\varepsilon}{6} + 5\frac{\varepsilon}{10} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Therefore, as $|x f(x) - 10| < \varepsilon$, we have that

$$\lim_{x\to 2} x f(x) = 10.$$

∎

* we can do this because $\lim_{x\to 2} f(x) = 5$.