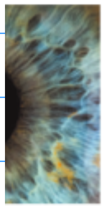
4 Applications of Differentiation



Homework 15 (quiz 15 on Thursday)
Section 4.2#19,20,21,22

19–20 Show that the equation has exactly one real root.

19. $2x + \cos x = 0$ **20.** $x^3 + e^x = 0$

let $f(x) = 2x + \cos x$

then $f'(x) = 2 - \sin x$ since $|\sin x| \leq 1$ then

$f'(x) = 2 - \sin x \geq 2 - 1 \geq 1$

so f is always increasing..

so f has at most one root

To see that there is a root note

$f(-1) = -2 + \cos(-1) < 0$

$f(1) = 2 + \cos(1) > 0$

So the root must be in the interval $[-1, 1]$ by the intermediate value property of continuous functions.

L'Hospital's Rule

Suppose f, g are differentiable functions with

$f(a)=0, \quad g(a)=0, \quad g'(a) \neq 0$

and the also the derivatives f' and g' are continuous.

assumption.

Since f and g are differentiable, they are continuous, so

$$\lim_{x\to a} f(x) = f(a) = 0 \qquad \lim_{x\to a} f'(x) = f'(a)$$

$$\lim_{x\to a} g(x) = g(a) = 0 \qquad \lim_{x\to a} g'(x) = g'(a) \neq 0$$

Question: What is $\lim_{x\to a} \frac{f(x)}{g(x)}$?

Can't use usual limit laws...

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0}$$

doesn't make sense, the limit law the requires denominator to be non-zero.

This limit may not be infinity.. also not likely equal 1, but could be 1, we just don't know.

Example

$$\lim_{x\to 0} \frac{1}{x^2} = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} x^2} = \frac{1}{0^+}$$

this is ∞

New question: What is $\lim_{x\to a} \frac{f'(x)}{g'(x)}$?

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to a} f'(x)}{\lim g'(x)} = \frac{f'(a)}{g'(a)}$$

assumption is $g'(a) \neq 0$

Can this limit be related to the one we can't compute?

YES!

L'Hospital's Rule

Why

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}{\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}}$$

hypothesis

$f(a)=0$

$g(a)=0$

limit law again

$$= \frac{\lim_{h\to 0}\frac{f(a+h)}{h}}{\lim_{h\to 0}\frac{g(a+h)}{h}} = \lim_{h\to 0}\left(\frac{f(a+h)}{h}\Big/\frac{g(a+h)}{h}\right)$$

$$= \lim_{h\to 0}\frac{f(a+h)}{g(a+h)} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

let $x = a+h$

then $x\to a$ as $h\to 0$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x)\neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

8. $\lim_{x\to 3} \frac{x-3}{x^2-9}$ try plugging in $\frac{3-3}{3^2-9} = \frac{0}{0}$ ← indeterminant form

General strategy for limits is simply to see what's happening

write an equivalent expression that is continuous at the limit

$$\frac{x-3}{x^2-9} = \frac{\cancel{x-3}}{(\cancel{x-3})(x+3)} = \frac{1}{x+3}$$

$$\lim_{x\to 3} \frac{x-3}{x^2-9} = \lim_{x\to 3} \frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$$

continuous at $x=3$ so can plug in to find the limit.

By L'Hospital's rule

$$\lim_{x\to 3} \frac{\overbrace{\boxed{x-3}}^{f(x)}}{\underbrace{\boxed{x^2-9}}_{g(x)}} = \underbrace{\lim_{x\to 3} \frac{f'(x)}{g'(x)} = \lim_{x\to 3} \frac{\frac{d}{dx}(x-3)}{\frac{d}{dx}(x^2-9)}} = \lim_{x\to 3} \frac{1}{2x} = \frac{1}{6}$$

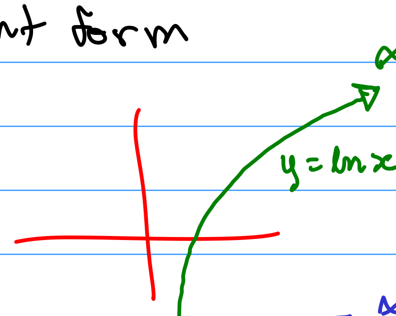
different way of finding the same answer.

Another example:

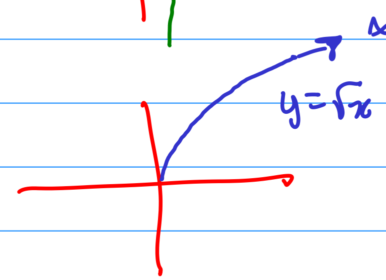
Check if indeterminant form

19. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

$$\lim_{x\to\infty} \ln x = \infty$$



$$\lim_{x\to\infty} \sqrt{x} = \infty$$



Since $\frac{\infty}{\infty}$ is indeterminant I can use L'Hospital's rule ...

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^{1/2}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2x^{1/2}}}$$

$$= \lim_{x\to\infty} \frac{1}{x} \cdot \frac{2x^{1/2}}{1} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$$

Example...

67. $\lim_{x\to 0^+} (1+\sin 3x)^{1/x} = \lim_{x\to 0^+} e^{\frac{1}{x}\ln(1+\sin 3x)}$

do algebra to write as a fraction somewhere..

exponential is continuous

$$= e^{\lim_{x\to 0^+} \frac{1}{x}\ln(1+\sin 3x)} = e^3$$

is this indeterminant?

$\lim_{x\to 0^+} \ln(1+\sin 3x) = \ln 1 = 0$

$\lim_{x\to 0^+} x = 0$

Now find $\lim_{x\to 0^+} \frac{\ln(1+\sin 3x)}{x}$

$0/0$ ✓

$$\lim_{x\to 0^+} \frac{\ln(1+\sin 3x)}{x} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(1+\sin 3x)}{\frac{d}{dx}x}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+\sin 3x}(\cos 3x)3}{1} = \frac{1}{1+\sin 0}(\cos 0)3 = 3.$$