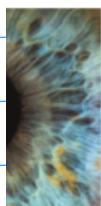


4

Applications of Differentiation



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← Today

← more graphing later

Homework 15 (quiz 15 on Thursday)
 Section 4.2#19,20,21,22

19–20 Show that the equation has exactly one real root.

→ **19.** $2x + \cos x = 0$

20. $x^3 + e^x = 0$

let $f(x) = 2x + \cos x$

then $f'(x) = 2 - \sin x$ since $|\sin x| \leq 1$ then

$$f'(x) = 2 - \sin x \geq 2 - 1 \geq 1$$

so f is always increasing.

so f has at most one root

To see that there is a root note

$$f(-1) = -2 + \cos(-1) < 0$$

$$f(1) = 2 + \cos(1) > 0$$

So the root must be in the interval $[-1, 1]$ by the intermediate value property of continuous functions.

L'Hospital's Rule

Suppose f, g are differentiable functions with

$$f(a)=0, g(a) \neq 0, g'(a) \neq 0$$

and the also the derivatives f' and g' are continuous.
assumption.

Since f and g are differentiable, they are continuous, so

$$\lim_{x \rightarrow a} f(x) = f(a) = 0$$

$$\lim_{x \rightarrow a} f'(x) = f'(a)$$

$$\lim_{x \rightarrow a} g(x) = g(a) \neq 0$$

$$\lim_{x \rightarrow a} g'(x) = g'(a) \neq 0$$

Question: What is $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$?

Can't use usual limit laws...

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{0}{0}$$

doesn't make sense, this limit law requires the denominator to be non-zero.

Example

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \frac{\lim_{x \rightarrow 0} 1}{\lim_{x \rightarrow 0} x^2} = \frac{1}{0^+}$$

This limit may not be infinity... also not likely equal 1, but could be 1, we just don't know.
this is so

New question: What is $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$?

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)} = \frac{f'(a)}{g'(a)}$$

assumption is $g'(a) \neq 0$

Can this limit be related to the one we can't compute?
YES!

L'Hospital's Rule

Why

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}}{\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}}$$

$$= \frac{\lim_{h \rightarrow 0} \frac{f(a+h)}{h}}{\lim_{h \rightarrow 0} \frac{g(a+h)}{h}} \stackrel{\text{limit law again}}{\approx} \lim_{h \rightarrow 0} \left(\frac{\frac{f(a+h)}{h}}{\frac{g(a+h)}{h}} \right)$$

hypothesis

$$\begin{aligned} f(a) &\approx 0 \\ g(a) &= 0 \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{f(a+h)}{g(a+h)} = \lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

let $x = a+h$

then $x \rightarrow a$ as $h \rightarrow 0$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

or that $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞ .) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

8. $\lim_{x \rightarrow 3} \frac{x-3}{x^2-9}$ try plugging in $\frac{3-3}{3^2-9} = \frac{0}{0}$ indeterminant form

General strategy for limits is simply to see what's happening

write an equivalent expression that is continuous at the limit

$$\frac{x-3}{x^2-9} = \frac{x-3}{(x-3)(x+3)} = \frac{1}{x+3}$$

$$\lim_{x \rightarrow 3} \frac{x-3}{x^2-9} = \lim_{x \rightarrow 3} \frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$$

continuous at $x=3$ so can plug in to find the limit.

By L'Hospital's rule

$$\lim_{x \rightarrow 3} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 3} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 3} \frac{\frac{d}{dx}(x-3)}{\frac{d}{dx}(x^2-9)} = \lim_{x \rightarrow 3} \frac{1}{2x} = \frac{1}{6}$$

different way of finding the same answer.

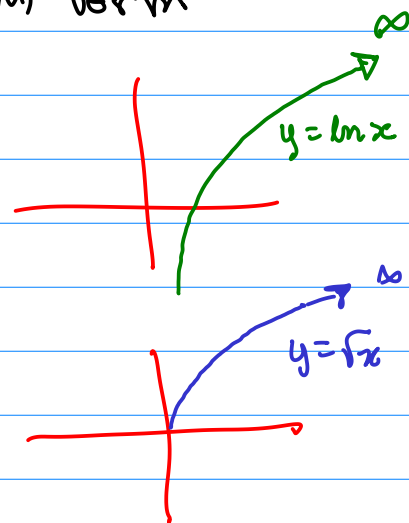
Another example:

Check if indeterminate form

19. $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$

$$\lim_{x \rightarrow \infty} \ln x = \infty$$

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$



Since $\frac{\infty}{\infty}$ is indeterminate I can use L'Hospital's rule...

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} &\approx \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x^{1/2}} = \lim_{x \rightarrow \infty} \frac{1/x}{\frac{1}{2} x^{-1/2}} = \lim_{x \rightarrow \infty} \frac{1/x}{\frac{1}{2} x^{1/2}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \frac{2 x^{1/2}}{1} = \lim_{x \rightarrow \infty} \frac{2}{x^{1/2}} = 0 \end{aligned}$$

Example ...

$$67. \lim_{x \rightarrow 0^+} (1 + \sin 3x)^{1/x} = \lim_{x \rightarrow 0^+} e^{\frac{1}{x} \ln(1 + \sin 3x)}$$

↑
do algebra to
write as a fraction
somewhere...

↑
exponential is
continuous

$$= e^{\lim_{x \rightarrow 0^+} \frac{1}{x} \ln(1 + \sin 3x)} = e^3$$

Now find $\lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 3x)}{x}$

is this indeterminate?

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln(1 + \sin 3x) &= \ln 1 = 0 \\ \lim_{x \rightarrow 0^+} x &= 0 \end{aligned}$$

0/0 ☒

$$\lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 3x)}{x} \approx \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(1 + \sin 3x)}{\frac{d}{dx} x}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{1 + \sin 3x} (\cos 3x) 3 = \frac{1}{1 + \sin 0} (\cos 0) 3 = 3$$