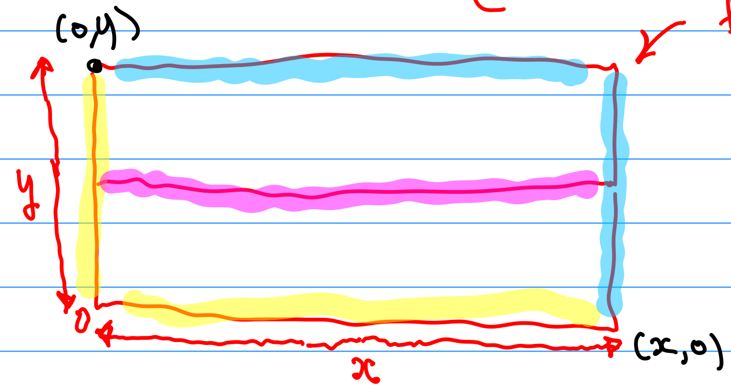
$A = 1.5$

13. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

C = cost



fenced off field

not only decide the optimal direction (horizontal or vertical in picture) but we vary the dimensions of the field.

Remark, since we are free to draw the rectangle like ▯ or like ▭ then there is no need to worry about whether the dividing fence is one way or the other.

$A = xy$ formula for area of rectangle.

$y = \frac{A}{x}$

$C = P(y + x + y + x + x) = P(3x + 2y)$

P = unit price of fence = \$/ft

- assume P is the same for all the fence used.

minimize C. subject to $x \geq 0$, $y \geq 0$, $A = 1.5$

Algebra to simplify & Calculus to find the minimum.

$$C(x) = P \cdot \left(3x + 2\frac{A}{x}\right)$$

Note cost is proportional to length of the fence so we could just as well minimize the length.

$$L(x) = 3x + 2\frac{A}{x}$$

$$L'(x) = 3 - \frac{2A}{x^2} = 0$$

$$3x^2 = 2A \qquad \boxed{x = \pm\sqrt{\frac{2A}{3}}}$$

$$\underbrace{x \geq 0, \; y \geq 0}_{[0, \infty)}$$

only the positive critical value is in the domain of the function

What is the minimum?

Check end points of the interval and check the critical values.

very narrow

$$\lim_{x \to 0^+} L(x) = \lim_{x \to 0^+} 3x + 2\frac{A}{x} = \frac{A}{0^+} = \infty$$

very wide

$$\lim_{x \to \infty} L(x) = \lim_{x \to \infty} 3x + 2\frac{A}{x} = 3 \cdot \infty = \infty$$

Bad Ideas when minimizing length of the fence ...

$$L'(x) = 3 - \frac{2A}{x^2} = \frac{3x^2 - 2A}{x^2} = 3\,\frac{x^2 - \frac{2}{3}A}{x^2}$$

$$= \frac{3}{x^2}\left(x - \sqrt{\tfrac{2}{3}A}\right)\left(x + \sqrt{\tfrac{2}{3}A}\right)$$

max

where a minimum is ..

not in the domain

		$-\sqrt{\frac{2}{3}A}$		0		$+\sqrt{\frac{2}{3}A}$	
$\left(x - \sqrt{\frac{2}{3}A}\right)$	$-$		$-$		$-$		$+$
$\left(x + \sqrt{\frac{2}{3}A}\right)$	$-$		$+$		$+$		$+$
$\frac{3}{x^2}$	$+$		$+$		$+$		$+$
$L'(x)$	$+$		$-$		$-$		$+$
$L(x)$	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

min

$$x = \sqrt{\tfrac{2}{3}A} = \sqrt{\tfrac{2}{3}\, 1.5 \text{ million ft}^2}$$

$$= \sqrt{\text{million ft}^2} = \sqrt{1000^2 \text{ ft}^2} = 1000 \text{ ft}$$

$$y = \frac{A}{x} = \frac{1.5 \text{ million ft}^2}{1000 \text{ ft}} = 1.5 \text{ thousand ft.}$$

$$= 1500 \text{ ft}$$

To minimize cost

... fence a field with dimensions 1000 ft by 1500 ft and divide in half parallel to the shorter side.

For next time consider

14. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.