Suppose $f'(x) = 1$ what could $f(x)$ be?

$$\frac{d}{dx} x = 1 \qquad f(x) = x + c$$

antiderivative

antiderivative unique up to a constant

Suppose $f'(x) = x$ what could $f(x)$ be?

$$\frac{d}{dx} x^2 = 2x$$

$$\frac{d}{dx}\frac{1}{2}x^2 = \frac{\frac{d}{dx}x^2}{2} = \frac{2x}{2} = x \qquad f(x) = \frac{1}{2}x^2 + c$$

antiderivative

Suppose $f'(x) = x^n$ what could $f(x)$ be?

$$\frac{d}{dx} x^{n+1} = (n+1)x^n$$

$$\frac{d}{dx}\frac{1}{n+1}x^{n+1} = \frac{\frac{d}{dx}x^{n+1}}{n+1} = \frac{(n+1)x^n}{n+1} = x^n \qquad \text{so } f(x) = \frac{1}{n+1}x^{n+1} + c$$

Suppose $f'(x) = \ln x$ what could $f(x)$ be?

Idea: product rule

$$\frac{d}{dx} x\ln x = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

on other side

not a derivative

$$\left(\frac{d}{dx} x\ln x\right) - 1 = \ln x$$

to write 1 as a derivative $\frac{d}{dx}x$

factor out

$$\left(\frac{d}{dx} x\ln x\right) - \frac{d}{dx}x = \ln x$$

$$\frac{d}{dx}(x\ln x - x) = \ln x$$

so $f(x) = x\ln x - x + c$

Suppose $f'(x) = \frac{1}{1+x^2}$ what could $f(x)$ be?

$$\frac{d}{dx}\tan x = \frac{1}{1+x^2}$$

$f(x) = \mathrm{tax} + c$

Suppose $f'(x) = \frac{1}{4+x^2}$ what could $f(x)$ be?

$$\frac{d}{dx}\frac{1}{4}\tan x = \frac{1}{4}\frac{d}{dx}\tan x = \frac{1}{4}\cdot\frac{1}{1+x^2} = \frac{1}{4+4x^2} = \frac{1}{4+u^2}$$

? substitution $u = 2x$

$$\frac{d}{dx}\tan 2x = \frac{1}{1+(2x)^2}\frac{d}{dx}2x = \frac{2}{1+4x^2}$$

correct guess

$$\frac{d}{dx}\tan\frac{x}{2} = \frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{d}{dx}\frac{x}{2} = \frac{1/2}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{4}{4} = \frac{2}{4+x^2}$$

$2 = \sqrt{4}$

$$\frac{d}{dx}\frac{1}{2}\tan\frac{x}{2} = \frac{\frac{d}{dx}\tan\frac{x}{2}}{2} = \frac{\frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{d}{dx}\frac{x}{2}}{2} = \frac{\frac{1/2}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{4}{4}}{2} = \frac{\frac{2}{4+x^2}}{2} = \frac{1}{4+x^2}$$

Therefore $f(x) = \frac{1}{2}\tan\frac{x}{2} + c$

Suppose $f'(x) = \frac{1}{a+x^2}$ what could $f(x)$ be?

$$\frac{d}{dx}\frac{1}{\sqrt{a}}\tan\frac{x}{\sqrt{a}} = \frac{1}{\sqrt{a}}\frac{1}{1+\left(\frac{x}{\sqrt{a}}\right)^2}\frac{d}{dx}\frac{x}{\sqrt{a}} = \frac{1}{\sqrt{a}}\frac{1}{1+\left(\frac{x}{\sqrt{a}}\right)^2}\cdot\frac{1}{\sqrt{a}} = \frac{1}{a}\frac{1}{1+\frac{1}{a}x^2} = \frac{1}{a+x^2}$$

Therefore $f(x) = \frac{1}{\sqrt{a}}\tan\frac{x}{\sqrt{a}} + c$

if $a > 0$ this works well
if $a < 0$ then $\sqrt{a}$ is imaginary and what does that mean?

If $a = 0$ then what?

Suppose $f'(x) = \frac{1}{x^2} = x^{-2}$ what could $f(x)$ be?

$n = -2$

recall

Suppose $f'(x) = x^n$ what could $f(x)$ be?

$$f(x) = \frac{1}{n+1}x^{n+1} + c$$

$$f(x) = \frac{1}{-2+1}x^{-2+1} + c = -x^{-1} + c = \boxed{\frac{-1}{x} + c}$$

Assume $a < 0$

Suppose $f'(x) = \dfrac{1}{a + x^2}$ what could $f(x)$ be?

too difficult...

Simpler, try $a = -1$

Suppose $f'(x) = \dfrac{1}{-1 + x^2}$ what could $f(x)$ be?

what goes here ...

$$\frac{1}{-1+x^2} = \frac{1}{\underbrace{(x-1)(x+1)}} = \frac{?}{x-1} + \frac{?}{x+1}$$

View this as the common denominator of two simpler fractions

want to be the same...
solve for A and B so they are the same.

$$\frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1) + B(x-1)}{(x-1)(x+1)} = \frac{x(A+B) + A - B}{(x-1)(x+1)}$$

$$x\underbrace{(A+B)} + \underbrace{A-B} = 1$$

$A + B = 0$ $\qquad$ $A - B = 1$

$A = -B$

$2A = 1$ $\qquad$ so $A = \frac{1}{2}$ $\qquad$ $B = -\frac{1}{2}$

$$\frac{1}{-1+x^2} = \frac{1}{\underbrace{(x-1)(x+1)}} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

Suppose $f'(x) = \dfrac{1/2}{x-1} + \dfrac{-1/2}{x+1}$ what could $f(x)$ be?

$$\frac{d}{dx}\frac{1}{2}\left[\ln(x-1) - \ln(x+1)\right] = \frac{1/2}{x-1} - \frac{1/2}{x+1}$$

Therefore $f(x) = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + c$

Suppose $f'(x) = \dfrac{1}{-1+x^2}$

Therefore $f(x) = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + c$

What about... $a = -b^2$

Suppose $f'(x) = \dfrac{1}{x^2 - b^2} = \dfrac{1}{(x-b)(x+b)} = \dfrac{A}{x-b} + \dfrac{B}{x+b}$

Therefore $f(x) = A\ln(x-b) + B\ln(x+b) + c$

arbitrary constant

need to solve for A and B to finish the problem so

$$A(x+b) + B(x-b) = 1$$

$$(A+B)x + (A-B)b = 1$$

$A = -B$ and $(A-B)b = 1$