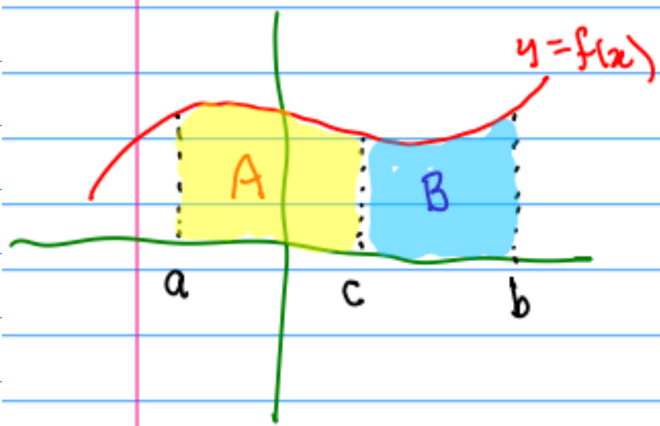
Areas Add

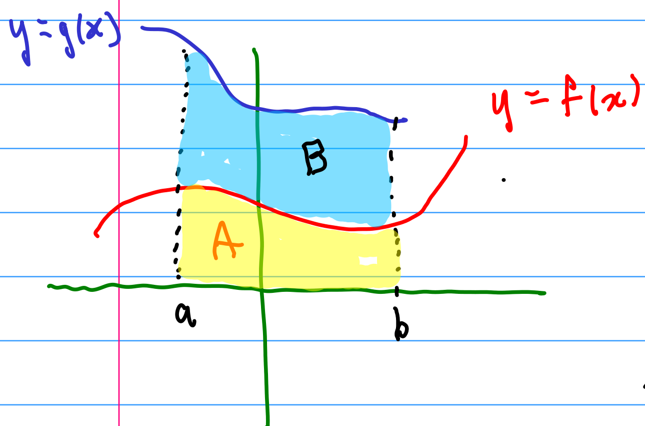


$$A = \int_a^c f(x)\,dx$$

$$B = \int_c^b f(x)\,dx$$

result:

$$\int_a^b f(x)\,dx = A + B = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



$$A = \int_a^b f(x)\,dx$$

area above $f(x)$ and below $g(x)$

$$B = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

Thus

$$\int_a^b g(x)\,dx = A + B.$$

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \,(= a), x_1, x_2, \ldots, x_n \,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

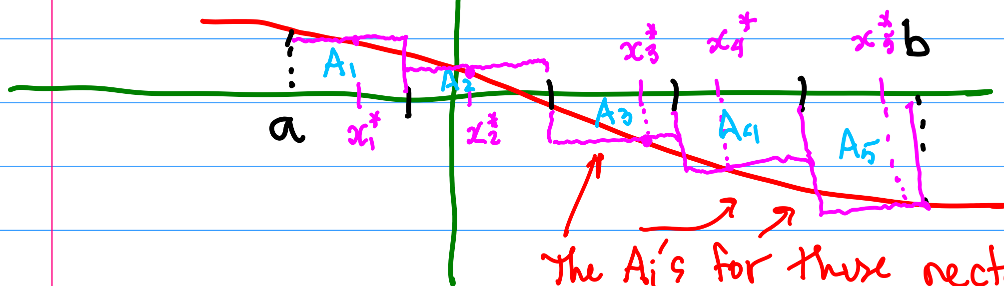
This formula make sense even if f is negative.

Σ means a sum...

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

$n=5$, $\Delta x = \frac{b-a}{n}$

$A_1 = f(x_1^*)\Delta x \quad \dots \quad A_i = f(x_i^*)\Delta x$

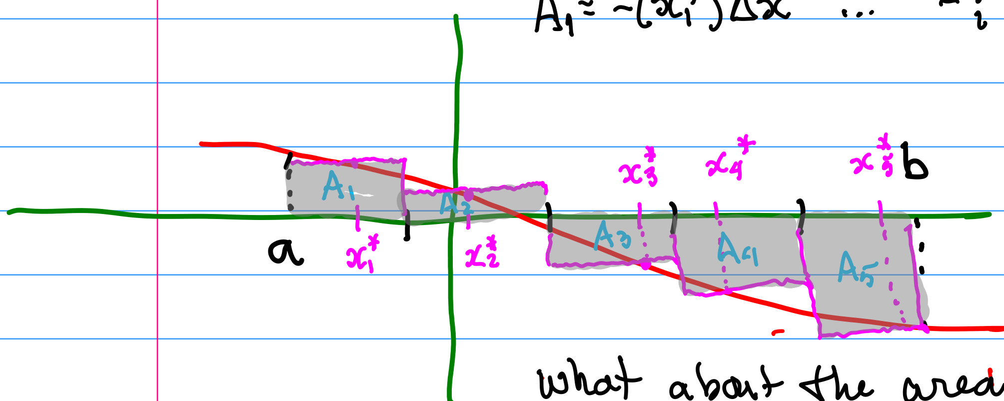


The A_i's for these rectangles is negative
area below ground is counted negative...

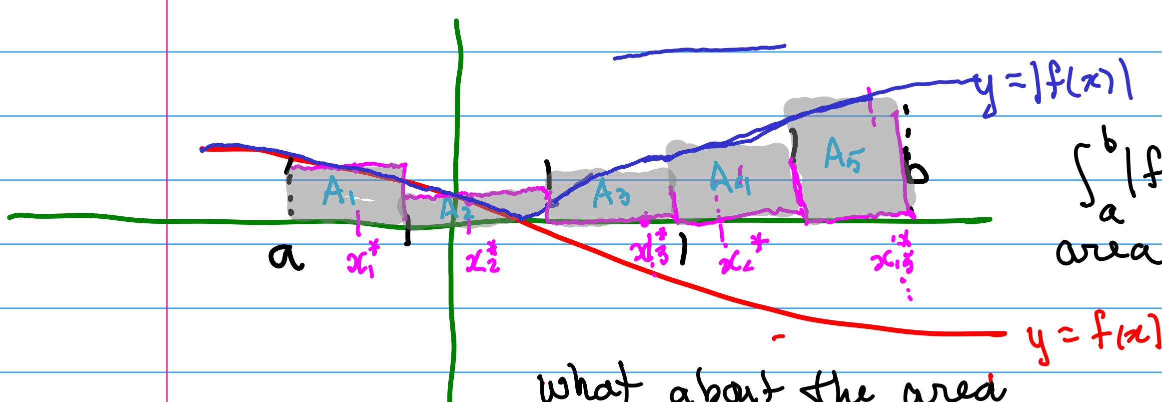
$$\int_a^b f(x)\,dx \approx \underbrace{A_1+A_2}_{\text{positive}} + \underbrace{A_3+A_4+A_5}_{\text{negative..}}$$

no longer interpret as area under a curve....

$A_1 = -(x_1^*)\Delta x \quad \dots \quad -_i =$



what about the area shown not making the rectangles underground negative.

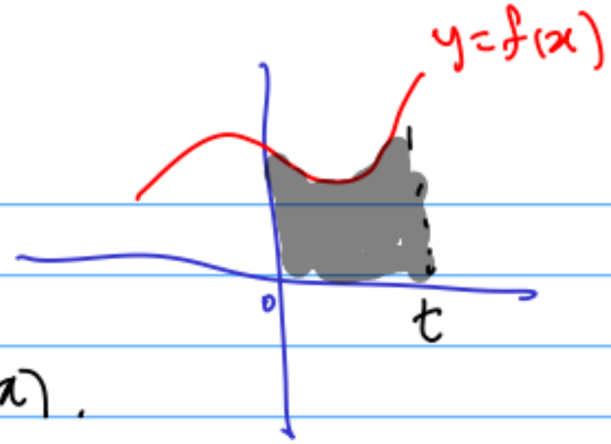


$\int_a^b |f(x)|\,dx$
area between curve and the x-axis.

what about the area

Note, since $|f(x)| \geq f(x)$ then $\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$
(all the areas count positive)

$$F(t) = \int_0^t f(x)\,dx$$



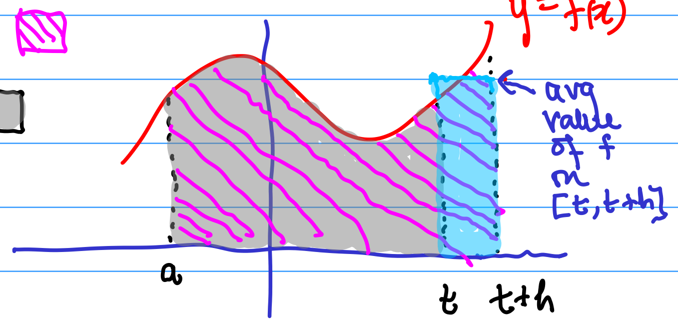
we want to show $\dfrac{dF(x)}{dx} = f(x)$.

Could start the area defining the function anywhere

$$F(t) = \int_a^t f(x)\,dx$$

F(t+h) =

F(t) =



assume h>0

Let's differentiate F(t) to see what we get..

$$F'(t) = \lim_{h\to 0} \frac{F(t+h) - F(t)}{h} = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(x)\,dx$$

this is the average value of f over an interval of length h.

Areas add

$$\underbrace{\int_a^t f(x)\,dx}_{F(t)} + \int_t^{t+h} f(x)\,dx = \underbrace{\int_a^{t+h} f(x)\,dx}_{F(t+h)}$$

assume h>0

Claim

after the length of averaging interval goes to zero, just get f(t)

$$F'(t) = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(x)\,dx = f(t)$$

assume h>0

geometrically clear what this limit is, what about using equations, inequalities and algebra?

Show that I can choose h small enough

Since $f(t)$ doesn't depend on x then $f(t) = \frac{1}{h}\int_t^{t+h} f(t)\,dx$

x

assume $h>0$

$$\left|\frac{1}{h}\int_t^{t+h} f(x)\,dx - f(t)\right| = \left|\frac{1}{h}\int_t^{t+h} f(x)\,dx - \frac{1}{h}\int_t^{t+h} f(t)\,dx\right|$$

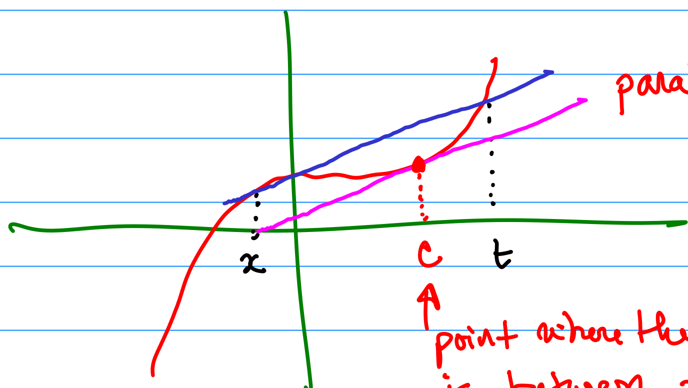
since $f(t)$ does not depend on x

Areas add (or subtract).

$$= \left|\frac{1}{h}\int_t^{t+h} \big(f(x) - f(t)\big)\,dx\right|$$

for any x between t and $t+h$ what is. Mean value theorem

$f(x) - f(t) = f'(c)\,(x-t)$ for some c between x and t.



point where the parallel translation is tangent is between x and t

Note c depends on t and x in this picture (note t is constant...)

assume $h>0$

$$\left|\frac{1}{h}\int_t^{t+h} f(x)\,dx - f(t)\right| = \left|\frac{1}{h}\int_t^{t+h}\big(f(x)-f(t)\big)\,dx\right|$$

$$\leq \frac{1}{h}\int_t^{t+h} |f(x) - f(t)|\,dx = \frac{1}{h}\int_t^{t+h} |f'(c(x))(x-t)|\,dx$$

let $B = \max\limits_{c\in[x,x+1]} |f'(c)|$

$$\frac{1}{h}\int_t^{t+h} |f'(c(x))(x-t)|\,dx \leq \frac{1}{h}\int_t^{t+h} B\,\underbrace{|x-t|}_{\text{less than } h}\,dx$$

$$\leq \frac{1}{\cancel{h}}\int_t^{t+h} B\cancel{h}\,dx = \int_t^{t+h} B\,dx = hB$$

$$\lim_{h\to 0}\left|\frac{1}{h}\int_t^{t+h} f(x)\,dx - f(t)\right| \leq \lim_{h\to 0} hB = 0$$

thus

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(x)\,dx = f(t).$$