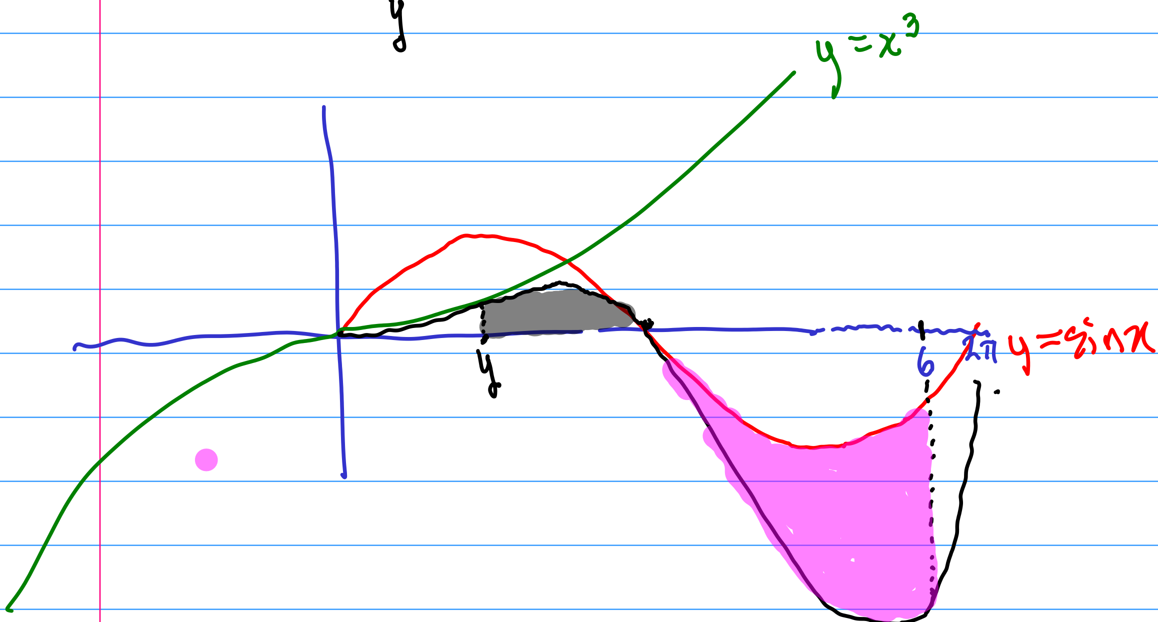
So far we have The Fundamental Theorem of Calculus (part 1)

If $F(t) = \int_a^t f(x)\,dx$ then $F'(t) = f(t)$

Examples:

$F(t) = \int_3^t \sin x\,dx$ then $F'(t) = \sin t$

$F(y) = \int_y^6 x^3 \sin x\,dx$ then $F'(y) = ?$



2 Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

This formula make sense even if f is negative.

Σ means a sum...

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

If $a > b$ then this limit can also be computed, but what does it mean?

Note that if $a > b$ then $\Delta x = \frac{b-a}{n}$ is negative.
these are the same rectangles as if we took $\frac{a-b}{n}$ but
they are counted as negative area...

$$F(y) = \int_y^6 x^3 \sin x \, dx = -\int_6^y x^3 \sin x \, dx$$

$$F'(y) = -\frac{d}{dy}\int_6^y x^3 \sin x \, dx = -y^3 \sin y.$$

Exercise. #14 Example

$$h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1} \, dz \qquad \text{what is } h'(x) =$$

let $F(t) = \int_1^t \frac{z^2}{z^4+1} \, dz$ then $F'(t) = \frac{t^2}{t^4+1}$

Then $h(x) = F(\sqrt{x})$ so

$$h'(x) = F'(\sqrt{x}) \frac{d}{dx}\sqrt{x} = F'(\sqrt{x}) \frac{1}{2} x^{-1/2} = \frac{(\sqrt{x})^{\cancel{2}}}{(\sqrt{x})^4 + 1} \frac{1}{2\cancel{\sqrt{x}}}$$

$$h'(x) = \frac{\sqrt{x}}{2(x^2+1)}$$

Also

If $F(t) = \int_1^t \frac{1}{x} \, dx$ then $F'(t) = \frac{1}{t}$

I also know that $\frac{d}{dt} \ln t = \frac{1}{t}$ so that means

both $\ln t$ and $F(t)$ are antiderivates of $\frac{1}{t}$.
Therefore $\ln t$ and $F(t)$ differ by a constant.

Fundamental Theorem of Calculus (part 2).

<u>Claim</u>: If $\boxed{F(t)}$ is an antiderivative of $f(t)$ then $\int_a^b f(t)\,dt = F(b) - F(a)$.

Define $g(t) = \int_a^t f(x)\,dx$. Then by part 1 of the Fundamental theorem of calculus $g'(t) = f(t)$.

Therefore $F(t)$ and $g(t)$ are antiderivatives of $f(t)$, So they differ by a constant

$$F(t) = g(t) + c \quad \text{the constant,} \qquad g(t) = F(t) - c$$

By the definition of g we have.

$$g(b) = \int_a^b f(x)\,dx \quad \text{also} \quad g(a) = \int_a^a f(x)\,dx = 0$$

just subtract 0.

Then
$$\int_a^b f(x)\,dx = g(b) = g(b) - g(a)$$
$$= F(b) - c - (F(a) - c) = F(b) - F(a).$$

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§5.3 // **23.** $\int_1^9 \sqrt{x}\,dx$ what is the antiderivative of $f(x) = \sqrt{x} = x^{1/2}$

$$\frac{d}{dx} x^{3/2} = \frac{3}{2} x^{1/2} \qquad 3/2 \quad 3/2 \qquad \text{Then } \frac{d}{dx}\frac{2}{3}x^{3/2} = x^{1/2}$$

Therefore $F(x) = \frac{2}{3}x^{3/2}$

$$\int_1^9 \sqrt{x}\,dx = F(9) - F(1) = \frac{2}{3}9^{3/2} - \frac{2}{3}1^{3/2} = 18 - \frac{2}{3} = 17\tfrac{1}{3}.$$

Shorthand notation

$$\int_1^9 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_{x=1}^{9} = \frac{2}{3}9^{3/2} - \frac{2}{3}1^{3/2} = 18 - \frac{2}{3} = 17\frac{1}{3}.$$

sometimes "$x=$" is omitted here when it's clear what the variable is.

Example

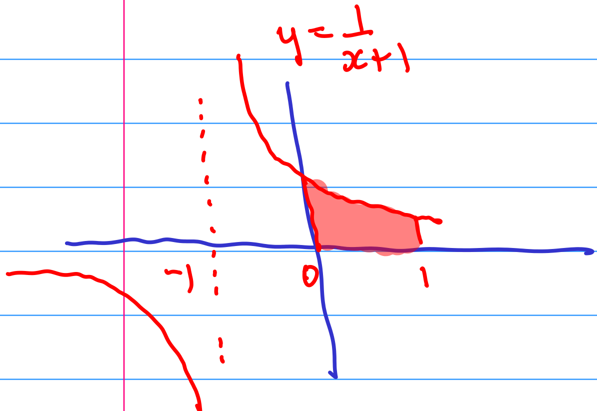
$$\int_0^1 \frac{x}{x+1}\,dx = \int_0^1 \left(1+\frac{1}{x+1}\right)dx = \int_0^1 1\,dx + \int_0^1 \frac{1}{x+1}\,dx$$

$$\begin{array}{r} 1 \\ x+1 \,\big)\overline{\,x\,} \\ \underline{x+1} \\ -1 \end{array}$$ ← remainder.

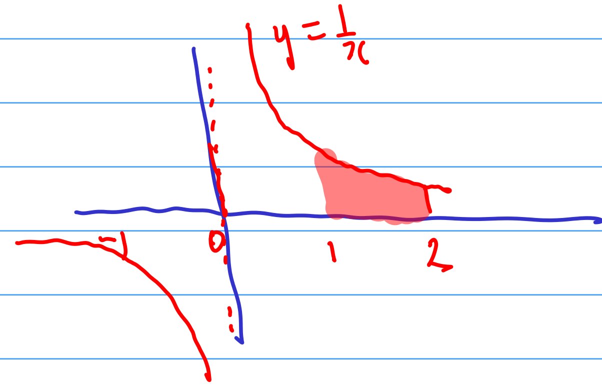
means $\dfrac{x}{x+1} = 1 + \dfrac{1}{x+1}$

In pieces

$$\int_0^1 1\,dx = 1 \qquad \text{also} \qquad \int_0^1 \frac{1}{x+1}\,dx = \int_1^2 \frac{1}{x}\,dx$$



Same area



$$\int_1^2 \frac{1}{x}\,dx = \int_1^2 x^{-1}\,dx = \ln x\Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

$$\int_0^1 \frac{x}{x+1}\,dx = 1 + \ln 2,$$