Sums of Positive Terms

Following are a selections of tests to determine whether a sum $\sum_{n=1}^{\infty} a_n$ of positive terms with $0 < a_n < \infty$ for $n = 1, 2, 3, \ldots$ is finite or infinite.

Tail Test: Let N be a positive integer. Then

$$\sum_{n=1}^{\infty} a_n < \infty \qquad \text{if and only if} \qquad \sum_{n=N}^{\infty} a_n < \infty.$$

Comparison Test: If $a_n \le b_n$ for all $n \ge N$ then

$$\sum_{n=1}^{\infty} b_n < \infty \qquad \text{implies} \qquad \sum_{n=1}^{\infty} a_n < \infty.$$

Limit Comparison Test: Suppose $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$ where $0 < L < \infty$. Then

$$\sum_{n=1}^{\infty} a_n < \infty \qquad \text{if and only if} \qquad \sum_{n=1}^{\infty} b_n < \infty.$$

Moreover, if $L = 0$ then

$$\sum_{n=1}^{\infty} b_n < \infty \qquad \text{implies} \qquad \sum_{n=1}^{\infty} a_n < \infty$$

and if $L = \infty$ then

$$\sum_{n=1}^{\infty} a_n < \infty \qquad \text{implies} \qquad \sum_{n=1}^{\infty} b_n < \infty.$$

Integral Comparison Test: Let $f(x)$ be a decreasing function such that $a_n = f(n)$ for $n = 1, 2, 3, \ldots$. Then

$$\sum_{n=1}^{\infty} a_n < \infty \qquad \text{if and only if} \qquad \int_1^{\infty} f(x)\,dx < \infty.$$

Ratio Test: Suppose $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$ and $L \ne 1$. Then

$$L < 1 \quad \text{implies} \quad \sum_{n=1}^{\infty} a_n < \infty \qquad \text{and} \qquad L > 1 \quad \text{implies} \quad \sum_{n=1}^{\infty} a_n = \infty.$$

Root Test: Suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ and $L \ne 1$. Then

$$L < 1 \quad \text{implies} \quad \sum_{n=1}^{\infty} a_n < \infty \qquad \text{and} \qquad L > 1 \quad \text{implies} \quad \sum_{n=1}^{\infty} a_n = \infty.$$