Math 182 Honors Very Long Quiz 7 Version A ~ Key ~

1. Find the following indefinite integrals:

(i) $\int \frac{1}{x^2 - x - 2}\,dx = \int \frac{1}{(x-2)(x+1)}\,dx = \int \left(\frac{A}{x-2} + \frac{B}{x+1}\right)dx$

$= A\ln|x-2| + B\ln|x+1| + C$

Solve for A and B

$A(x+1) + B(x-2) = 1$

$A+B=0 \qquad A-2B=1$

$B=-A \qquad 3A=1$

$A=\frac{1}{3}$

$B=-\frac{1}{3}$

Final Answer:

$\frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$

(ii) $\int x^2 e^{-x}\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx$

by parts $u=x^2 \quad du=2x\,dx$, $dv=e^{-x}dx \quad v=-e^{-x}$

by parts again $u=2x \quad du=2dx$, $dv=e^{-x}dx \quad v=-e^{-x}$

$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$

Final answer:

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

(iii) $\int \arctan\sqrt{x}\,dx = x\arctan\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}\,dx$

by parts $u=\arctan\sqrt{x} \quad du=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\,dx$, $dv=dx \quad v=x$

Now solve $\frac{1}{2}\int \frac{\sqrt{x}}{1+x}\,dx = \int \frac{p^2}{1+p^2}\,dx = \int\left(\frac{1+p^2}{1+p^2} - \frac{1}{1+p^2}\right)dp =$

$p=\sqrt{x} \quad p^2=x \quad 2p\,dp=dx$

$= \int\left(1-\frac{1}{1+p^2}\right)dp = p - \arctan p + C = \sqrt{x} - \arctan\sqrt{x} + C$

Final answer:

$x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C$

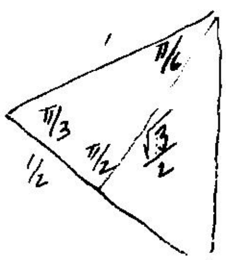
Math 182 Honors Very Long Quiz 7 Version A

2. Find the following definite integrals:

(i) $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$

(ii) $\int_0^{\pi/6} \cos(2x)\,dx = \frac{1}{2}\sin 2x\Big|_0^{\pi/6} = \frac{1}{2}\sin \pi/3 - \frac{1}{2}\sin 0$

$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - 0 = \boxed{\frac{\sqrt{3}}{4}}$



(iii) $\int_1^{e^2} \ln x\,dx = x\ln x - x\Big|_1^{e^2} = e^2\ln e^2 - e^2 - 1\ln 1 + 1$

$= 2e^2 - e^2 + 1 = \boxed{e^2+1}$

Indefinite integral (by parts):

$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}dx = x\ln x - x$

$u = \ln x,\quad du = \frac{1}{x}dx$

$dv = dx,\quad v = x$

Math 182 Honors Very Long Quiz 7 Version A

3. State Taylor's Theorem with the integral form of the remainder term.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$$+ \frac{(x-a)^n}{n!}f^{(n)}(a) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

4. State Taylor's forumula with remainder term expanded about $a = 0$ for the functions

(i) e^x $= \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \int_0^x \frac{(x-t)^n}{n!} e^t\,dt}$

(ii) $\ln(1+x)$ $= \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t}\,dt}$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}$$

$$+ (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \frac{(-1)^{n+1}x^{n+1}}{1+x}$$

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(1 - t + t^2 - \cdots + (-1)^n t^n + \frac{(-1)^{n+1}t^{n+1}}{1+t}\right)dt$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^n x^{n+1}}{n+1} + (-1)^{n+1}\int_0^x \frac{t^{n+1}}{1+t}\,dt \quad \text{set}$$

5. Use Taylor's series to compute

(i) $\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$.

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$e^{x^2} = \cancel{1} + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$\cos x = \cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^{x^2} - \cos x = \frac{3}{2}x^2 + \left(\frac{1}{2} - \frac{1}{24}\right)x^4 + \cdots$$

Therefore:

$$\frac{e^{x^2} - \cos x}{x^2} = \frac{3}{2} + \left(\frac{1}{2} - \frac{1}{24}\right)x^2 + \cdots \longrightarrow \frac{3}{2} \quad \text{as } x \to 0$$

(ii) $\lim_{x\to 0} \frac{\log(1+x^2) - x^2\cos x}{x^6}$.

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\log(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots$$

$$x^2 \cos x = \cancel{x^2} - \frac{x^4}{2!} + \frac{x^6}{4!} - \cdots$$

$$\log(1+x^2) - x^2\cos x = \left(\frac{1}{3} - \frac{1}{24}\right)x^6 + \cdots$$

$$= \frac{7}{24}x^6 + \cdots$$

Therefore:

$$\frac{\log(1+x^2) - x^2\cos x}{x^6} = \frac{7}{24} + \cdots \longrightarrow \frac{7}{24} \quad \text{as } x \to 0$$

Math 182 Honors Very Long Quiz 7 Version A

6. Use the error term in Taylor's Theorem to determine a value for m such that

$$\sum_{k=0}^{m} \frac{(-1)^k\left(\frac{1}{3}\right)^{2k+1}}{(2k+1)!}$$

approximates $\sin\frac{1}{3}$ to within 10^{-13}. To minimize needless computation you may use the table at the bottom of the page.

The remainder term is

$$\left| \int_0^{1/3} \frac{\left(\frac{1}{3}-t\right)^{2m+2}}{(2m+2)!} \cos t \, dt \right| \le \int_0^{1/3} \frac{\left(\frac{1}{3}-t\right)^{2m+2}}{(2m+2)!} \, dt$$

$$= -\left.\frac{\left(\frac{1}{3}-t\right)^{2m+3}}{(2m+3)!}\right|_0^{1/3} = \frac{\left(\frac{1}{3}\right)^{2m+3}}{(2m+3)!}$$

$$= \frac{1}{3^{2m+3}(2m+3)!} \le 10^{-13}$$

Thus need to find m so $3^{2m+3}(2m+3)! \ge 10^{13}$

This product needs 14 digits

So $2m+3 \geq 12$

or $2m \geq 9$

or $\boxed{m=5}$

n	3^n	$n!$	#digits
1	3	1	1
2	9	2	2
3	27	6	3
4	81	24	4
5	243	120	5
6	729	720	6
7	2,187	5,040	8
8	6,561	40,320	9
9	19,683	362,880	10
10	59,049	3,628,800	12
11	177,147	39,916,800	13
→ 12	531,441	479,001,600	15
13	1,594,323	6,227,020,800	
14	4,782,969	87,178,291,200	

Math 182 Honors Very Long Quiz 7 Version A

7. Use the integral test to determine whether the following infinite series converge.

(i) $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ diverges, since

$$\int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty} = \ln\infty - 0 = \infty$$

So the series does <u>not</u> converge.

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$ converges, since

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \int_1^{\infty} x^{-3}\,dx = \frac{1}{-2} x^{-2} \Big|_1^{\infty}$$

$$= \frac{1}{-2x^2} \Big|_1^{\infty} = 0 + \frac{1}{2} < \infty$$

So this series converges.

(iii) $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} < \infty$ converges, since

$$\int_1^{\infty} \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\,dx = 2\int_1^{\infty} e^{-u}\,du = -2e^{-u} \Big|_1^{\infty}$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$= \frac{-2}{e^u} \Big|_1^{\infty} = 0 + \frac{2}{e^1} < \infty$$

So this series converges.