Math 182 Exam 1 Version B

Key

1. Find the following derivatives:

(i) $\frac{d}{dx}\cosh(2x+7)$

$= 2\sinh(2x+7)$

(ii) $\frac{d}{dx}\frac{x^2}{3+\arctan x}$

$$= \frac{2x(3+\arctan x) - x^2\frac{1}{1+x^2}}{(3+\arctan x)^2}$$

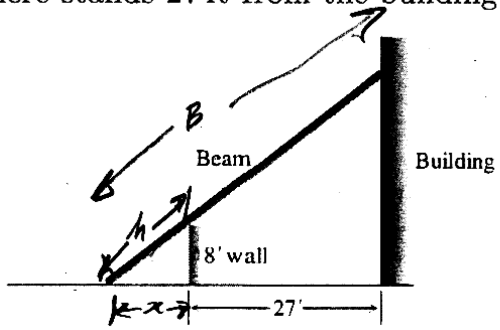
(iii) $\frac{d}{dx}|\cos(x)|^3 = 3|\cos x|^2 \frac{\cos x}{|\cos x|}(-\sin x)$

$= -3|\cos x|(\cos x)(\sin x)$

(iv) $\frac{d}{dx}(\ln(1+x^2))^x = [\ln(1+x^2)]^x \frac{d}{dx}(x\ln(\ln(1+x^2)))$

$$= [\ln(1+x^2)]^x\left(\ln(\ln(1+x^2)) + x\frac{1}{\ln(1+x^2)}\cdot\frac{1}{1+x^2}\cdot 2x\right)$$

2. The 8 ft wall shown here stands 27 ft from the building.



B = length of beam.
h = hypotenuse of little triangle
x = leg of little triangle.

Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

Pythagorean theorem for little triangle:

$$x^2 + 8^2 = h^2 \qquad h = \sqrt{8^2 + x^2}$$

Similar triangles:

$$\frac{h}{x} = \frac{B}{27 + x}$$

Thus

$$B = \frac{27+x}{x} h = \frac{(27+x)}{x}\sqrt{8^2+x^2}$$

$$= (27x^{-1}+1)\sqrt{8^2+x^2}$$

$$\frac{dB}{dx} = (-27x^{-2})\sqrt{8^2+x^2} + (27x^{-1}+1)\frac{x}{\sqrt{8^2+x^2}} = 0$$

$$-27x^{-2}(8^2+x^2) + (27x^{-1}+1)x = 0$$

$$-27(8^2+x^2) + (27+x)x^2 = 0$$

Therefore $\quad x^3 = 8^2 \cdot 27 = 2^6 3^3 \;;\; x = 4 \cdot 3 = 12.$

$$B = \frac{27+12}{12}\sqrt{8^2+12^2} = \frac{\overset{13}{39}}{3 \cdot 4} 4\sqrt{2^2+3^2} = 13\sqrt{13}.$$

The minimum length is $13\sqrt{13}$ feet.

3. State the Mean Value Theorem for derivatives.

Suppose f is continuous on $[a,b]$ and differentiable on (a,b) then there exists c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

4. State the Fundamental Theorem of Calculus Part I.

Suppose f is continuous on $[a,b]$. Then $F(t) = \int_a^t f(x)\,dx$ is continuous on $[a,b]$ differentiable on (a,b) and

$$\frac{dF(t)}{dt} = \frac{d}{dt}\int_a^t f(x)\,dx = f(t) \text{ for all } t \text{ in } (a,b)$$

5. State the Fundamental Theorem of Calculus Part II.

Suppose f is continuous on $[a,b]$ with antiderivative F on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

6. Explain how to use the trapezoidal rule to approximate $\int_a^b f(x)\,dx$.

Given an integer n define $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$ and $y_k = f(x_k)$ for $k = 0, 1, 2, \ldots, n$. Then

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

Choose n large enough so the approximation is good.

7. Solve the following indefinite integrals:

(i) $\int \sin(2x)\sin(3x)\,dx$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$

$= \int \frac{1}{2}(\cos x - \cos 5x)\,dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$

(ii) $\int \frac{x}{x^2-3x}\,dx = \int \frac{x}{x(x-3)}\,dx = \int \frac{1}{x-3}\,dx$

$= \ln|x-3| + C$

(iii) $\int \frac{1}{x^2+5}\,dx = \frac{1}{5}\int \frac{1}{1+\left(\frac{x}{\sqrt{5}}\right)^2}\,dx$

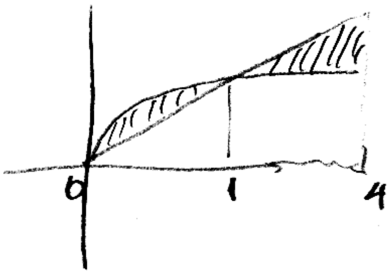
$u = \frac{x}{\sqrt{5}}$; $du = \frac{dx}{\sqrt{5}}$

$= \frac{1}{\sqrt{5}}\int \frac{1}{1+u^2}\,du = \frac{1}{\sqrt{5}}\arctan u + C$

$= \frac{1}{\sqrt{5}}\arctan\frac{x}{\sqrt{5}} + C.$

8. Solve the following definite integrals:

(i) $\int_0^4 |x-\sqrt{x}|\,dx = \int_0^1 (\sqrt{x}-x)\,dx + \int_1^4 (x-\sqrt{x})\,dx$

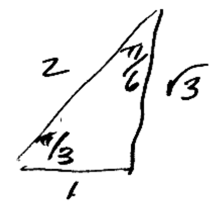


$= \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2\right)\Big|_0^1 + \left(\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right)\Big|_1^4$

$= \frac{2}{3} - \frac{1}{2} + \frac{16}{2} - \frac{2}{3}\cdot 8 - \frac{1}{2} + \frac{2}{3}$

$= \frac{-12}{3} + \frac{14}{2} = -4 + 7 = 3.$

(ii) $\int_0^1 \sqrt{4-x^2}\,dx = 2\int_0^1 \sqrt{1-\left(\frac{x}{2}\right)^2}\,dx$



$\sin u = \frac{x}{2} \qquad 2\cos u\,du = dx$

$= 4\int_{\arcsin 0}^{\arcsin \frac{1}{2}} \sqrt{1-\sin^2 u}\,\cos u\,du = 4\int_0^{\pi/6} \cos^2 u\,du = 2\int_0^{\pi/6} (1+\cos 2u)\,du$

$= 2\frac{\pi}{6} + (\sin 2u)\Big|_0^{\pi/6} = \frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

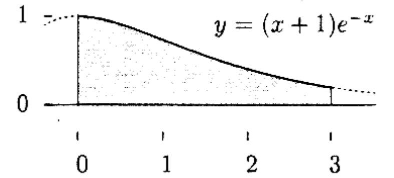
(iii) $\int_0^{\pi/6} \tan(2x)\,dx = \int_6^{\pi/6} \frac{\sin 2x}{\cos 2x}\,dx$

$u = \cos 2x \qquad du = -2\sin 2x\,du$

$= \frac{-1}{2}\int_{\cos 0}^{\cos \pi/3} \frac{1}{u}\,du = \frac{1}{2}\int_{1/2}^{1} \frac{1}{u}\,du = \frac{1}{2}\ln u\Big|_{1/2}^{1}$

$= -\frac{1}{2}\ln\frac{1}{2} = \frac{\ln 2}{2}.$

9. Find the volume generated by revolving the area under the curve $y = (x+1)e^{-x}$ between $x = 0$ and $x = 3$ about the x-axis.



$$V = \pi \int_0^3 y^2\,dx = \pi \int_0^3 (x+1)^2 e^{-2x}\,dx$$

$$u = (x+1)^2 \qquad du = 2(x+1)\,dx$$

$$dv = e^{-2x}\,dx \qquad v = -\tfrac{1}{2}e^{-2x}$$

$$= -\frac{\pi}{2}(x+1)^2 e^{-2x}\Big|_0^3 + \pi\int_0^3 (x+1)e^{-2x}\,dx$$

$$p = x+1 \qquad dp = dx$$

$$dq = e^{-2x}\,dx \qquad q = -\tfrac{1}{2}e^{-2x}$$

$$= -\frac{\pi}{2}(16e^{-6}-1) + -\frac{\pi}{2}(x+1)e^{-2x}\Big|_0^3 + \frac{\pi}{2}\int_0^3 e^{-2x}\,dx$$

$$= -\frac{\pi}{2}(16e^{-6}-1+4e^{-6}-1) + -\frac{\pi}{4}e^{-2x}\Big|_0^3 = \frac{\pi}{4}(5-41e^{-6})$$

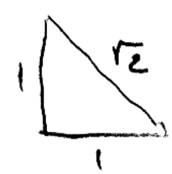
$$\approx 3.8471717\ldots$$

10. Find the length of the curve given by $y = \ln(\sec x)$ between $x = 0$ and $x = \pi/4$.

$$y = \ln\sec x = -\ln(\cos x) \qquad \frac{dy}{dx} = -\frac{1}{\cos x}\cdot(-\sin x) = \tan x$$

$$L = \int_0^{\pi/4} \sqrt{\tan^2 x + 1}\,dx = \int_0^{\pi/4}\sqrt{\sec^2 x}\,dx = \int_0^{\pi/4}\frac{1}{\cos x}\,dx$$

$$= \int_0^{\pi/4}\frac{\cos x}{1-\sin^2 x}\,dx = \int_0^{1/\sqrt{2}}\frac{1}{1-u^2}\,du$$



$$u = \sin x \qquad du = \cos x\,dx$$

$$= \frac{1}{2}\int_0^{1/\sqrt{2}}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)du = \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right)\Big|_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\ln\frac{1+u}{1-u}\Big|_0^{1/\sqrt{2}} = \frac{1}{2}\ln\left(\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}\right) = \frac{1}{2}\ln\frac{(1+1/\sqrt{2})^2}{1-1/2}$$

$$= \ln\left(\frac{1+1/\sqrt{2}}{1/\sqrt{2}}\right) = \ln(1+\sqrt{2})$$

$$\approx 0.8813735\ldots$$

11. Let $f(x) = \ln x$ and find the maximum of $|f''(x)|$ on the interval $[1, 3]$.

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$|f''(x)| = \frac{1}{x^2}$$

$$\begin{cases} |f''(1)| = 1 \\ |f''(3)| = \frac{1}{9} \end{cases}$$

maximum of $|f''(x)|$ on $[1, 3]$ is $\boxed{1}$

12. Consider approximating $\int_1^3 \ln x \, dx$ using the trapezoid method. Use the bound

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}$$

to determine a value for n which guarantees the error would be less than 10^{-4} but do not actually compute the approximation.

$$\frac{1 \cdot 2^3}{12 n^2} < 10^{-4} \qquad n^2 > \frac{2^3}{12} 10^4 = \frac{2}{3} 10^4$$

Thus $n > 100\sqrt{\frac{2}{3}} \approx 81.6496\ldots$

The value $n = 82$ is sufficient to guarantee the error is less than 10^{-4}.