Taylor series for $f(x) = (1+x)^\alpha$ when $a=0$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots +$$

$$+ \frac{(x-a)^n}{n!}f^{(n)}(a) + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

Differentiating

$f(x) = (1+x)^\alpha$ $\qquad$ $f(0) = 1$

$f'(x) = \alpha(1+x)^{\alpha-1}$ $\qquad$ $f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$ $\qquad$ $f''(0) = \alpha(\alpha-1)$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$ $\qquad$ $f'''(0) = \alpha(\alpha-1)(\alpha-2)$

$\vdots$ $\qquad$ $\vdots$

$f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$ $\qquad$ $f^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)$

$f^{(n+1)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)(1+x)^{\alpha-n-1}$

So

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots +$$

$$+ \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n + \int_0^x \frac{(x-t)^n}{n!}\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)(1+t)^{\alpha-n-1}\,dt$$

$$= \sum_{k=0}^{n} \binom{\alpha}{k} x^k + \int_0^x \binom{\alpha}{n+1}(n+1)(x-t)^n(1+t)^{\alpha-n-1}\,dt.$$

We now consider conditions under which the remainder term tends to zero.

First note that

$$\binom{\alpha}{n+1} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-M+1)(\alpha-M)(\alpha-M-1)\cdots(\alpha-n)}{1\cdot 2\cdot 3 \quad \cdots \quad M \cdot (M+1)\cdot(M+2)\cdots(n+1)}$$

and since $\displaystyle\lim_{k\to\infty} \frac{\alpha-k}{k+1} = -1$

then for every $\varepsilon > 0$ there is M large enough so that $k \geq M$ implies

$$\left| \frac{\alpha-k}{k+1} - (-1) \right| < \varepsilon.$$

Therefore

$$\left|\frac{\alpha-k}{k+1}\right| = \left|\frac{\alpha-k}{k+1} - (-1) + (-1)\right| \leq \left|\frac{\alpha-k}{k+1} - (-1)\right| + 1 < 1+\varepsilon$$

implies

$$\left|\binom{\alpha}{n+1}\right| \leq \left|\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-M+1)}{1\ 2\ 3\cdots M}\right| (1+\varepsilon)^{n-M+1}$$

$$= \left|\binom{\alpha}{M}\right| (1+\varepsilon)^{n-M+1}$$

We now show the remainder term

$$R_n = \int_0^x \binom{\alpha}{n+1}(n+1)(x-t)^n (1+t)^{\alpha-n-1}\,dt \to 0$$

as $n \to \infty$ provided $|x| < 1$.

Case $0 < x < 1$

By the Weighted Mean Value Theorem for Integrals

$$R_n = \int_0^x \binom{\alpha}{n+1}(n+1)(x-t)^n(1+t)^{\alpha-n-1}\,dt$$

$$= \binom{\alpha}{n+1}(1+\xi)^{\alpha-n-1}\int_0^x (n+1)(x-t)^n\,dt$$

$$= \binom{\alpha}{n+1}x^{n+1}(1+\xi)^{\alpha-n-1}$$ for some ξ between 0 and x.

Choose $\varepsilon = \frac{1-x}{x}$. Then $\varepsilon x < 1-x$ so $(\varepsilon+1)x < 1$

so $x < \frac{1}{1+\varepsilon}$.

From the previous page, there is M large enough so that

$$\left|\binom{\alpha}{n+1}\right| \le \left|\binom{\alpha}{M}\right|(1+\varepsilon)^{n-M+1} = C(1+\varepsilon)^{n-M+1}.$$

Therefore

$$|R_n| \le C(1+\varepsilon)^{n-M+1}x^{n+1}(1+\xi)^{\alpha-n-1}$$

$$= C(1+\varepsilon)^{-M+1}x(1+\xi)^{\alpha-1}\left(\frac{(1+\varepsilon)x}{1+\xi}\right)^n$$

Since $\xi \geq 0$ then

$$0 < \frac{(1+\varepsilon)x}{1+\xi} \le (1+\varepsilon)x < 1$$

therefore $|R_n| \to 0$ as $n \to \infty$.

Case $-1 < x < 0$.

To avoid problems estimating $\frac{(1+\xi)x}{1+\xi}$ when ξ is negative we work directly with the integral.

$$R_n = \int_0^x \binom{\alpha}{n+1}(n+1)(x-t)^n(1+t)^{\alpha-n-1}\,dt$$

Let $u = -t$ so $du = -dt$. Then

$$\cdots = \int_0^{-x} \binom{\alpha}{n+1}(n+1)(x+u)^n(1-u)^{\alpha-n-1}\,du$$

Since $-x = |x|$ we have

$$|R_n| \le \int_0^{|x|} \left|\binom{\alpha}{n+1}\right|(n+1)|x+u|^n|1-u|^{\alpha-n-1}\,du$$

$$= \int_0^{|x|} \left|\binom{\alpha}{n+1}\right|(n+1)\left(\frac{|x|-u}{1-u}\right)^n(1-u)^{\alpha-1}\,du$$

$$= \left(\frac{|x|-\xi}{1-\xi}\right)^n \int_0^{|x|} \left|\binom{\alpha}{n+1}\right|(n+1)(1-u)^{\alpha-1}\,du$$

$$= \left(\frac{|x|-\xi}{1-\xi}\right)^n \left|\binom{\alpha}{n+1}\right| \frac{(n+1)}{\alpha}\left(-(1-u)^{\alpha}\right)\Big|_0^{|x|}$$

$$= \left(\frac{|x|-\xi}{1-\xi}\right)^n \left|\binom{\alpha}{n+1}\right| \cdot \frac{n+1}{\alpha}\left|1-(1-|x|)^{\alpha}\right|$$

for some ξ between 0 and $|x|$.

Now

$$\binom{\alpha}{n+1}\frac{n+1}{\alpha} = \frac{\alpha\cdot(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{1\cdot 2\cdot 3\cdot \quad n+1}\cdot\frac{n+1}{\alpha}$$

$$= \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{1 \quad 2 \quad \cdots \quad n} = \binom{\alpha-1}{n}$$

So by the same argument as before. For $\varepsilon = \frac{1-|x|}{|x|}$ there is M large enough so that

$$\left|\binom{\alpha-1}{n}\right| \leq \left|\binom{\alpha-1}{M-1}\right|(1+\varepsilon)^{n-M+1} = C\,(1+\varepsilon)^{n-M+1}$$

Therefore

$$|R_n| \leq C(1+\varepsilon)^{-M+1}\,|1-(1-|x|)^{\alpha}|\left(\frac{(1+\varepsilon)(|x|-\xi)}{1-\xi}\right)^{n}$$

Now $0 \leq \xi \leq |x| < \frac{1}{1+\varepsilon}$ implies

$$0 \leq \xi(1+\varepsilon) \leq |x|(1+\varepsilon) < 1$$

$$0 \leq (|x|-\xi)(1+\varepsilon) < 1-\xi(1+\varepsilon)$$

So

$$0 \leq \frac{(|x|-\xi)(1+\varepsilon)}{1-\xi} < \frac{1-\xi-\xi\varepsilon}{1-\xi} = 1-\frac{\xi\varepsilon}{1-\xi} \leq 1$$

Therefore $|R_n| \to 0$ as $n \to \infty$.

Combining both cases we obtain that, provided $|x| < 1$ then $|R_n| \to 0$ as $n \to \infty$.