Honors Math 182 Final Version A

1. State Taylor's formula using big-$\mathcal{O}$ for the remainder where $a = 0$ for the functions

(i) e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + O(x^{n+1})$$

(ii) $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+3})$$

(iii) $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+2})$$

(iv) $\log(1-x)$ $= \ln(1-x)$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + O(x^{n+1})$$

(v) $\arctan x$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + O(x^{2n+3})$$

(vi) $(1+x)^\alpha$

$$(1+x)^\alpha = 1 + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \cdots + \binom{\alpha}{n} x^n + O(x^{n+1})$$

2. Find the following derivatives:

(i) $\dfrac{d}{dx}|\sinh 2x|^2 = \dfrac{d}{dx}(\sinh 2x)^2$

$= 2(\sinh 2x)(\cosh 2x)2$

$= 4 \sinh 2x \cosh 2x$

$= 2 \sinh 4x$

(ii) $\dfrac{d}{dx}\dfrac{x}{1+e^x} = \dfrac{1\cdot(1+e^x) - xe^x}{(1+e^x)^2}$

$= \dfrac{1+(1-x)e^x}{(1+e^x)^2}$

(iii) $\dfrac{d}{dx}\arctan(1+x^2) = \dfrac{1}{1+(1+x^2)^2}\cdot 2x$

(iv) $\dfrac{d}{dx}(2+\sin x)^{\ln x} = (2+\sin x)^{\ln x}\dfrac{d}{dx}\left(\ln x\cdot\ln(2+\sin x)\right)$

$= (2+\sin x)^{\ln x}\left(\dfrac{\ln(2+\sin x)}{x} + \dfrac{\ln x}{2+\sin x}\cdot\cos x\right)$

3. Solve the following indefinite integrals:

(i) $\int \ln(3x)\,dx \;=\; \frac{1}{3}\int \ln u\,du = \frac{1}{3}(u\ln u - u)$

$u = 3x$

$du = 3dx$

$= \frac{1}{3}(3x\ln 3x - 3x)$

$= x\ln 3x - x + C$

(ii) $\int \frac{x^2+17}{x+3}\,dx \;=\; \int \left(x - 3 + \frac{26}{x+3}\right) dx$

$$\begin{array}{r} x-3 \\ x+3\,\overline{)\,x^2+17} \\ \underline{x^2+3x} \\ -3x+17 \\ \underline{-3x-9} \\ 26 \end{array}$$

$= \frac{x^2}{2} - 3x + 26\ln(x+3) + C$

(iii) $\int x\sin^2 x\,dx = -x\sin x\cos x + \int (x\cos^2 x + \sin x\cos x)\,dx$

$u = x\sin x \quad du = (x\cos x + \sin x)\,dx$

$dv = \sin x\,dx \quad v = -\cos x$

$= -x\sin x\cos x + \int x\cos^2 x\,dx + \frac{1}{2}\sin^2 x$

$= -x\sin x\cos x + \int x(1-\sin^2 x)\,dx + \frac{1}{2}\sin^2 x$

$2\int x\sin^2 x\,dx = -x\sin x\cos x + \int x\,dx + \frac{1}{2}\sin^2 x$

$\int x\sin^2 x\,dx = -\frac{x\sin x\cos x}{2} + \frac{x^2}{4} + \frac{1}{4}\sin^2 x + C$

(iv) $\int \frac{x}{\sqrt[3]{x+1}}\,dx \;=\; \int \frac{u-1}{u^{1/3}}\,du = \int (u^{2/3} - u^{-1/3})\,du$

$u = x+1 \quad du = dx \quad x = u-1$

$= \frac{u^{5/3}}{5/3} - \frac{u^{2/3}}{2/3} = \frac{3}{5}(x+1)^{5/3} - \frac{3}{2}(x+1)^{2/3} + C$

4. Solve the following definite integrals:

(i) $\displaystyle\int_0^1 \frac{1}{(x+1)(x-2)}\,dx = \int_0^1 \left(\frac{A}{x+1} + \frac{B}{x-2}\right)dx = A\ln|x+1| + B\ln|x-2| \Big|_0^1$

$= A\ln 2 - B\ln 2 = (A-B)\ln 2 = -\frac{2}{3}\ln 2$

$A(x-2) + B(x+1) = 1$

$(A+B)x + B - 2A = 1$

$A = -B \qquad 3B = 1$

$B = 1/3$

$A = -1/3$

(ii) $\displaystyle\int_{-4}^1 x\sqrt{x+8}\,dx = \int_2^3 (u^2-8)u \cdot 2u\,du$

$u = \sqrt{x+8}, \quad x = u^2 - 8, \quad dx = 2u\,du$

$= 2\int_2^3 (u^4 - 8u^2)\,du = 2\left(\frac{u^5}{5} - \frac{8}{3}u^3\right)\Big|_2^3$

$= 2u^3\left(\frac{3u^2-40}{15}\right)\Big|_2^3 = 2\,\frac{3^3(3\cdot 3^2 - 40) - 2^3(3\cdot 2^2 - 40)}{15}$

$= \frac{-254}{15}$

$$\begin{array}{rr} 40 & 40 \\ 27 & -12 \\ \hline 13 & 28 \\ 27 & 8 \\ \hline 91 & 224 \\ 26 & \\ \hline 351 & \\ 224 & \\ \hline 127 & \\ 2 & \\ \hline 254 & \end{array}$$

(iii) $\displaystyle\int_0^1 \frac{1}{1+e^x}\,dx$

$u = e^x \quad du = e^x dx = u\,dx$

$= \int_1^e \frac{1}{(1+u)u}\,du = \int_1^e \left(\frac{A}{1+u} + \frac{B}{u}\right)du = A\ln(1+u) + B\ln u \Big|_1^e$

$= A\ln(1+e) - A\ln 2 + B$

$= \ln 2 - \ln(1+e) + 1$

$Au + B(1+u) = 1$

$(A+B)u + B = 1$

$B = 1, \quad A = -1$

(iv) $\displaystyle\int_0^\pi |\sin 2x|\cos x\,dx = \int_0^{\pi/2} \sin 2x\cos x\,dx - \int_{\pi/2}^{\pi} \sin 2x\cos x\,dx$

$\int \sin 2x\cos x\,dx = 2\int \sin x\cos^2 x\,dx = -2\int u^2\,du = -\frac{2}{3}u^3 = -\frac{2}{3}\cos^3 x$

$u = \cos x \quad du = -\sin x\,dx$

$\ldots = -\frac{2}{3}\cos^3 x\Big|_0^{\pi/2} + \frac{2}{3}\cos^3 x\Big|_{\pi/2}^{\pi} = \frac{2}{3} - \frac{2}{3} = 0$

5. Use Taylor's series or L'Hôpital's rule to find the following limits if they exist.

(i) $\lim_{x\to 0} \frac{2xe^x + \ln(1-2x)}{x^3}$

$2xe^x = 2x(1+x+\frac{x^2}{2}+O(x^3)) = \cancel{2x} + \cancel{2x^2} + x^3 + O(x^4)$

$\ln(1-2x) = -2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} + O(x^4) = -2x - \cancel{2x^2} - \frac{8}{3}x^3 + O(x^4)$

$$= \lim_{x\to 0} \frac{x^3 - \frac{8}{3}x^3 + O(x^4)}{x^3} = 1 - \frac{8}{3} = -\frac{5}{3}$$

(ii) $\lim_{x\to 0} \frac{\ln(1+x^2) - x^2\cos x}{x^6}$

$x^2\cos x = x^2(1 - \frac{x^2}{2} + \frac{x^4}{4!} - O(x^6)) = \cancel{x^2} - \cancel{\frac{x^4}{2}} + \frac{x^6}{4!} + O(x^8)$

$\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} + O(u^4)$

$\ln(1+x^2) = \cancel{x^2} - \cancel{\frac{x^4}{2}} + \frac{x^6}{3} + O(x^8)$

$$= \lim_{x\to 0} \frac{-\frac{x^6}{4!} + \frac{x^6}{3} + O(x^8)}{x^6} = -\frac{1}{4!} + \frac{1}{3} = \frac{1}{3}\left(1 - \frac{1}{8}\right) = \frac{7}{24}$$

(iii) $\lim_{x\to 0} \frac{\ln(1-x^2) + x\arctan x}{x^4}$

$\ln(1-x^2) = -\cancel{x^2} - \frac{x^4}{2} + O(x^6)$

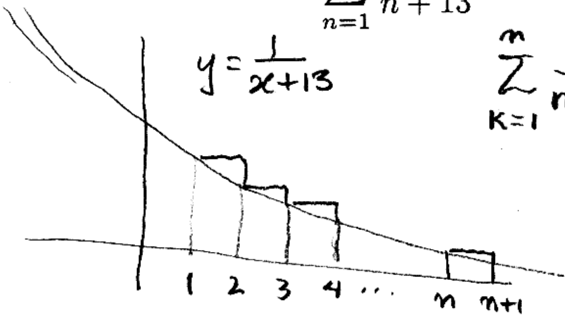
$x\arctan x = x(x - \frac{x^3}{3} + O(x^5)) = x^2 - \frac{x^4}{3} + O(x^6)$

$$= \lim_{x\to 0} \frac{-\frac{x^4}{2} - \frac{x^4}{3} + O(x^6)}{x^4} = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$$

(iv) $\lim_{x\to 0} \frac{\ln(1+x^2) + x^2\cos x}{x^6} = -\frac{7}{24}$

same as problem (ii).

6. Show that $\sum_{n=1}^{\infty} \frac{1}{n+13} = \infty$ by comparing the sum with a suitable integral.



$$\sum_{K=1}^{n} \frac{1}{n+13} \geq \int_1^{n+1} \frac{1}{x+13}\,dx = \ln(x+13)\Big|_1^{n+1}$$

$$= \ln(n+14) - \ln 14 \to \infty$$

as $n \to \infty$

Therefore the series is infinite.

7. Show that $\sum_{n=1}^{\infty} \frac{n^2}{2^n} < \infty$ by using the ratio test.

$$a_n = \frac{n^2}{2^n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \lim_{n\to\infty} \frac{n^2+2n+1}{n^2}$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{1}{2} < 1$$

So by the ratio test the series converges.

8. Give an example of a series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n > 0$ that converges.

Let $a_n = 1/2^n$.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{-1/2}{1+1/2} = \frac{-1/2}{3/2} = -\frac{1}{3}$$

This is a geometric series of the form

$$\sum_{n=1}^{\infty} p^n \text{ where } |p| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$$

So it converges

9. Give an example of a series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n > 0$ that diverges.

Let $a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

Then

$$\sum_{n=1}^{\infty} (-1)^n a_n = -1 + 2 - 1 + 2 - 1 + 2 - 1 + 2 - \cdots = \infty.$$

10. Find the arc length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 2$.

Let $x = f(t) = t$ so $y = g(t) = t^{3/2}$ where $0 \le t \le 2$.
Then $f'(t) = 1$ and $g'(t) = \frac{3}{2} t^{1/2}$.

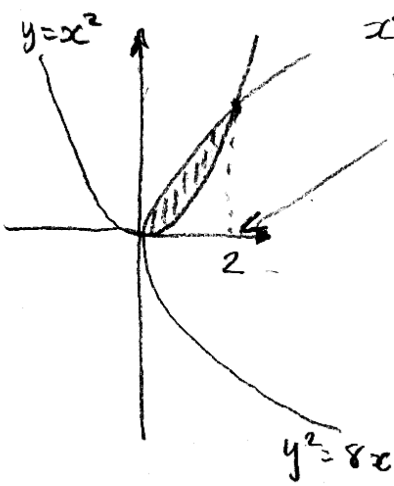
Therefore

$$L = \int_0^2 \sqrt{f'(t)^2 + g'(t)^2}\, dt = \int_0^2 \sqrt{1 + \frac{9}{4} t}\, dt = \int_1^{11/2} \frac{4}{9} \sqrt{u}\, du$$

$u = 1 + \frac{9}{4} t \qquad du = \frac{9}{4} dt$

$$= \frac{4}{9} \frac{u^{3/2}}{3/2} = \frac{8}{27}\left(\left(\frac{11}{2}\right)^{3/2} - 1\right)$$

11. Find the volume generated by revolving the region bounded by the curves $y = x^2$ and $y^2 = 8x$ around the x axis.



$x^4 = y^2 = 8x$
$x^3 = 8$
$x = 2$

$$V_x = \int_0^2 \pi(8x - x^4)\, dx$$
$$= \pi\left(4x^2 - \frac{x^5}{5}\right)\Big|_0^2$$
$$= \pi x^2 \left(\frac{20 - x^3}{5}\right)\Big|_0^2$$
$$= 4\pi \frac{20-8}{5} = \frac{48\pi}{5}$$

12. Find the curvature κ and radius of curvature ρ at the point $(e, 1)$ on the curve given by $(f(t), g(t))$ where $f(t) = e^t$ and $g(t) = \sin(\pi t/2)$ where $0 \le t \le 2$

$$\begin{cases} f(t_0) = e^{t_0} = e & \text{so } t_0 = 1 \\ g(t_0) = \sin(\pi t_0/2) = 1 & \text{so } t_0 = 1, 5, 9, \ldots \end{cases}$$

$f'(t) = e^t \quad f''(t) = e^t, \qquad f'(1) = e,\ f''(1) = e$

$g'(t) = \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right),\ g''(t) = -\frac{\pi^2}{4}\sin\left(\frac{\pi t}{2}\right), \quad g'(1) = 0,\ g''(1) = -\frac{\pi^2}{4}$

$$K = \frac{g''(1) f'(1) - g'(1) f''(1)}{(f'(1)^2 + g'(1)^2)^{3/2}} = \frac{-\frac{\pi^2}{4} \cdot e - 0 \cdot e}{(e^2 + 0^2)^{3/2}} = \frac{-\pi^2}{4e^2}$$

and $\rho = \frac{1}{|K|} = \frac{4e^2}{\pi^2}$.

13. Comparison with the geometric series yields

> **Theorem.** If there is N large enough such that $|a_n| < cq^n$ for some $q \in (0,1)$ and all $n \geq N$, then $\sum_{n=1}^{\infty} a_n$ converges.

Use this theorem to prove one of the following corollaries:

(i) Ratio Test. Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ where $L < 1$. Then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Root test. Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ where $L < 1$, Then $\sum_{n=1}^{\infty} a_n$ converges.

Root Test: Let $\varepsilon = \frac{1-L}{2}$. Then by hypothesis there is N so large that $n \geq N$ implies $|\sqrt[n]{|a_n|} - L| < \varepsilon$.

Then $\sqrt[n]{|a_n|} < L + \varepsilon = L + \frac{1-L}{2} = \frac{1+L}{2} = q < 1$ for $n \geq N$.

Taking n-th power of both sides

$$|a_n| < q^n \quad \text{for } n \geq N.$$

Applying the theorem with $c = 1$ gives that $\sum_{n=1}^{\infty} a_n$ converges.

Ratio Test. Let $\varepsilon = \frac{1-L}{2}$. Since $L<1$ then $\varepsilon > 0$.
By hypothesis there is N so large that $n \geq N$ implies
$$\left| \frac{|a_{n+1}|}{|a_n|} - L \right| < \varepsilon.$$

Then
$$\frac{|a_{n+1}|}{|a_n|} < L+\varepsilon = L + \frac{1-L}{2} = \frac{L+1}{2} = q \quad \text{for } n \geq N$$
where $q = \frac{L+1}{2} < 1$.

Consequently
$$|a_{n+1}| < q|a_n| \quad \text{for } n \geq N.$$

Let $c = \dfrac{|a_N|+1}{q^N}$.

Then
$$|a_N| = cq^N - 1 < cq^N$$
and
$$|a_{N+1}| < q|a_N| < cq^{N+1}$$
$$|a_{N+2}| < q|a_{N+1}| < cq^{N+2}$$
$$\vdots$$

So by induction $|a_n| < cq^n$ for $n \geq N$.

Applying the Theorem gives that $\sum_{n=1}^{\infty} a_n$ converges.

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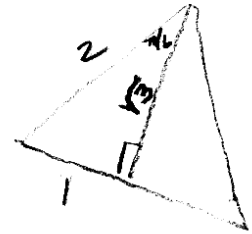

14. Consider the curve $(f(t), g(t))$ given by

$$f(t) = t\cos t \quad \text{and} \quad g(t) = \sqrt{t}\sin t \quad \text{where} \quad 0 \le t \le 2\pi.$$

Find the equation of the line tangent to the curve at the point $(f(\pi/6), g(\pi/6))$.

$f'(t) = \cos t - t\sin t \qquad f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}\cdot\frac{1}{2}$

$g'(t) = \frac{1}{2\sqrt{t}}\sin t + \sqrt{t}\cos t \qquad g'(\frac{\pi}{6}) = \sqrt{\frac{3}{2\pi}}\cdot\frac{1}{2} + \sqrt{\frac{\pi}{6}}\cdot\frac{\sqrt{3}}{2}$

$f(\frac{\pi}{6}) = \frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}, \quad g(\frac{\pi}{6}) = \sqrt{\frac{\pi}{6}}\cdot\frac{1}{2}, \quad m = \frac{g'(\frac{\pi}{6})}{f'(\frac{\pi}{6})} = \frac{\sqrt{\frac{3}{2\pi}} + \sqrt{\frac{\pi}{2}}}{\sqrt{3} - \frac{\pi}{6}}$

$$y - \frac{1}{2}\sqrt{\frac{\pi}{6}} = \frac{\sqrt{\frac{3}{2\pi}} + \sqrt{\frac{\pi}{2}}}{\sqrt{3} - \frac{\pi}{6}}\left(x - \frac{\pi}{4\sqrt{3}}\right)$$

15. Find the surface of revolution generated by revolving the arc $y = \frac{1}{4}x^4 + \frac{1}{8x^2}$ where $1 \le x \le 2$ about the y-axis. $f(t) = t, \quad g(t) = \frac{1}{4}t^4 + \frac{1}{8}t^{-2}, \quad 1 \le t \le 2$

$f'(t) = 1 \quad g'(t) = t^3 - \frac{1}{4}t^{-3}$

$$A_y = \int_1^2 2\pi f(t)\sqrt{f'(t)^2 + g'(t)^2}\,dt = \int_1^2 2\pi t\sqrt{1 + (t^3 - \tfrac{1}{4}t^{-3})^2}\,dt$$

$$= \int_1^2 2\pi t\sqrt{1 + t^6 - \tfrac{1}{2} + \tfrac{1}{16}t^{-6}}\,dt = \int_1^2 2\pi t\left(t^3 + \tfrac{1}{4}t^{-3}\right)dt$$

$$= 2\pi\left(\frac{t^5}{5} - \frac{1}{4}t^{-1}\right)\Big|_1^2 = 2\pi\left(\frac{2^5}{5} - \frac{1}{5} - \frac{1}{8} + \frac{1}{4}\right) = \pi\,\frac{248 + 5}{20} = \frac{253\pi}{20}$$

16. Find the area enclosed by the closed parametric curve $(f(t), g(t))$ given by $f(t) = \sin t$ and $g(t) = \cos t$ where $0 \le t \le 2\pi$.

$$A = \int_0^{2\pi} f(t)g'(t)\,dt = -\int_0^{2\pi}\sin^2 t\,dt = -2\int_0^{\pi}\sin^2 t\,dt$$

$$= -2\int_0^{\pi}\frac{1 - \cos 2t}{2}\,dt = -2\left(\frac{1}{2}\pi - \frac{\sin 2t}{4}\Big|_0^{\pi}\right) = -\pi$$

The signed area is negative since the area appears on the right side of the curve.

The enclosed area is π.
(Note this is the area of a circle of radius 1.)