Honors Math 182 Quiz 7 Version A

~ Key ~

1. State Taylor's Theorem.

Let f be an $n+1$ times continuously differentiable function. Then

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!}(x-a)^k f^{(k)}(a) + R_n$$

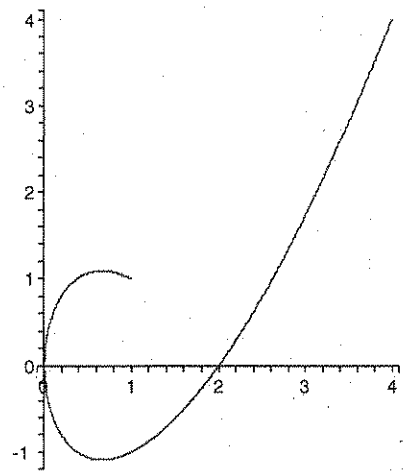
where

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between a and x.

2. Find the curvature κ and the radius of curvature ρ at the point $(0,0)$ on the curve given by $\big(f(t), g(t)\big)$ where $f(t) = t^2$, $g(t) = t^3 - 2t$ and $-1 \le t \le 2$.

$f(t) = t^2$	$f(0) = 0$
$f'(t) = 2t$	$f'(0) = 0$
$f''(t) = 2$	$f''(0) = 2$
$g(t) = t^3 - 2t$	$g(0) = 0$
$g'(t) = 3t^2 - 2$	$g'(0) = -2$
$g''(t) = 6t$	$g''(0) = 0$



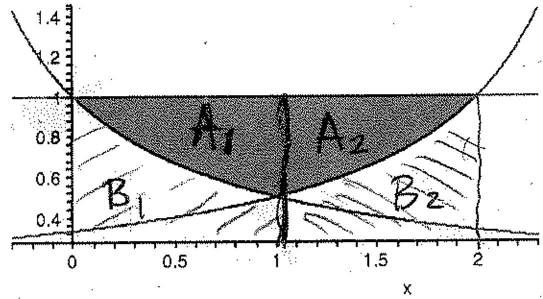
Therefore at the point $(0,0)$ we have $t = 0$ and

$$\kappa = \frac{f'(0)g''(0) - f''(0)g'(0)}{\left(f'(0)^2 + g'(0)^2\right)^{3/2}} = \frac{0\cdot 0 - 2(-2)}{\left(0^2 + (-2)^2\right)^{3/2}}$$

$$= \frac{4}{4^{3/2}} = \frac{4}{8} = \frac{1}{2}.$$

3. The area bounded by $y = 1/(x+1)$, $y = 1/(3-x)$ and $y = 1$ is depicted below.

(i) Find the volume formed by rotating this area about the x-axis.



By symmetry the volume is twice the volume of the region A_1 rotated about the x-axis.

The volume of A_1 rotated about the x axis can be given by subtracting the volume obtained from B_1 from the volume of the cylinder,

$$V_x = 2\left(\pi 1^2 \cdot 1 - \int_0^1 \frac{\pi}{(x+1)^2}\,dx\right) = 2\pi + \left.\frac{2\pi}{x+1}\right|_0^1$$

$$= 2\pi + \pi - 2\pi = \pi$$

(ii) Find the volume formed by rotating this area about the y-axis.

We add the volume formed by rotating A_1 about the y-axis the the volume formed by rotating A_2 about the y-axis.

This is the same as the volume of the cylinder minus the volume made by B_1 and B_2 rotated about the y-axis,

$$V_y = \pi 2^2 \cdot 1 - \int_0^1 2\pi x \frac{1}{x+1}\,dx - \int_1^2 2\pi x \frac{1}{3-x}\,dx$$

$$\int_0^1 \frac{x}{x+1}\,dx = \int_0^1 \left(1 - \frac{1}{x+1}\right)dx = x - \ln(x+1)\Big|_0^1 = 1 - \ln 2$$

$$\int_1^2 \frac{x}{3-x}\,dx = \int_1^2 \left(\frac{3}{3-x} - 1\right)dx = -3\ln(3-x) - x\Big|_1^2 = 3\ln 2 - 1$$

Therefore $V_y = 4\pi - 2\pi(2\ln 2) = 4\pi(1 - \ln 2)$.