Honors Math 182 Homework 5 Version A

1. Taylor's formula for $f(x) = \ln(1-x)$ when $a = 0$ is

$$\ln(1-x) = -\sum_{k=1}^{n} \frac{x^k}{k} - \int_0^x \frac{t^n}{1-t}\,dt.$$

(i) Given $|x| < 1$, the remainder may be estimated as

$$\left| \int_0^x \frac{t^n}{1-t}\,dt \right| \le \left| \int_0^x \frac{|t|^n}{1-|x|}\,dt \right| = \frac{1}{1-|x|} \int_0^{|x|} t^n\,dt = \frac{|x|^{n+1}}{(1-|x|)(n+1)}.$$

Use this estimate to show that if $|x| < 1$ then the remainder term tends to zero as $n \to \infty$.

(ii) Given $x = 1/2$ estimate how large n needs to be so the bound on the remainder is less than 0.5×10^{-4}.

(iii) Use the value of n found in Part (ii) to approximate $\ln(1/2)$.

2. Taylor's formula for $f(x) = \sin x$ when $a = 0$ is

$$\sin x = \sum_{l=0}^{m} (-1)^l \frac{x^{2l+1}}{(2l+1)!} + (-1)^{m+1} \int_0^x \frac{(x-t)^{2m+2}}{(2m+2)!} \cos t\,dt.$$

(i) Show for $x = 3$ that the remainder term tends to zero as $m \to \infty$.

(ii) Estimate how large m needs to be so the magnitude of the remainder is less than 0.5×10^{-4}.

(iii) Use the value of m found in Part (ii) to approximate $\sin 3$.

3. The trigonometric identity $\sin x = \sin(\pi - x)$ shows that

$$\sin x = \sum_{l=0}^{m} (-1)^l \frac{(\pi - x)^{2l+1}}{(2l+1)!} + (-1)^{m+1} \int_0^{\pi - x} \frac{(\pi - x - t)^{2m+2}}{(2m+2)!} \cos t\,dt.$$

(i) For $x = 3$ estimate how large m needs to be so the magnitude of the remainder is less than 0.5×10^{-4}.

(ii) Use the value of m found in Part (i) to approximate $\sin 3$.

4. Check the speed of convergence of the series approximation for $\sin 3$ found in Question 2 and the series approximation in Question 3 by computing the respective sums for values of $m = 1, 2, \ldots, 10$ and comparing each sum to the exact value of $\sin 3$.

5. Use Taylor series or L'Hôpital's rule to find the following limits if they exist.

(i) $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2}$

(ii) $\displaystyle\lim_{x\to 0^+} \frac{x^6}{\ln(1+x^2) - x^2\cos x}$

(iii) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x^3}$

(iv) $\displaystyle\lim_{x\to 0^-} \frac{\sin x - x\cos x}{x^3}$

(v) $\displaystyle\lim_{x\to 0} \frac{\ln(1-x^2) + x\arctan x}{x^4}$

(vi) $\displaystyle\lim_{x\to 0^-} \frac{1-\cos x}{\sin x}$

(vii) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x}{\cos x}$

(viii) $\displaystyle\lim_{x\to 0^+} \frac{xe^{-x^2} - \sin x}{2xe^x + \ln(1-2x)}$

(ix) $\displaystyle\lim_{x\to 0} \frac{1-\sqrt{1+4x^2}}{\sin^4 x}$

(x) $\displaystyle\lim_{x\to 0} \frac{x + (x-1)\ln(x+1)}{xe^x - \sin x}$