Math 285 Final Version A

1. Solve the initial value problem

$$ty' + 2y = \sin t \qquad \text{where} \qquad y(\pi/2) = 1.$$

2. Solve the initial value problem

$$y' = xy^3(1 + x^2)^{1/2} \qquad \text{where} \qquad y(0) = 1.$$

3. Find the critical points of the autonomous first-order differential equation

$$\frac{dy}{dx} = y^3 - 9y$$

Classify each critical point as asymptotically stable, unstable or semi-stable.

4. Solve the equation

$$y\,dx + (2x - ye^y)\,dy = 0$$

Hint: Make it exact by using an integrating factor $\mu = \mu(y)$.

5. State Euler's method for solving the ordinary differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$.

6. Given n let $h = (b - x_0)/n$ and $x_i = x_0 + hi$. The fourth-order Runge-Kutta method for solving $y' = f(x, y)$ such that $y(x_0) = y_0$ is given by

$$\begin{cases} k_1 = hf(x_i, y_i) \\ k_2 = hf(x_i + h/2, y_i + k_1/2) \\ k_3 = hf(x_i + h/2, y_i + k_2/2) \\ k_4 = hf(x_i + h, y_i + k_3) \\ y_{i+1} = y_i + (1/6)(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

Explain what it means in terms of the absolute error $|y(b) - y_n|$ and the step-size h that this method is fourth-order.

7. State the definition of the convolution $(f * g)(t)$ of two functions $f(t)$ and $g(t)$.

8. Let $f(t) = \sin t$ and $g(t) = \cos t$. Find the Laplace transform $\mathcal{L}\{f * g\}$.

9. Find a linear differential operator that annihilates $\cos(5x)$.

10. The variation of parameters formula to find a particular solution y_p is

$$y_p = -y_1 \int \frac{y_2 f(x)}{y_1 y_2' - y_1' y_2} dx + y_2 \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} dx$$

where y_1 and y_2 are solutions to the corresponding homogeneous problem. Use this formula to find a particular solution y_p to the linear inhomogeneous second-order ordinary differential equation

$$x^2 y'' - 2y = 3x^2 - 1, \qquad x > 0$$

with homogeneous solutions $y_1 = x^2$ and $y_2 = x^{-1}$.

11. Find the following Laplace transforms.

(i) $\mathcal{L}\{3t\}$

(ii) $\mathcal{L}\{2t^{-1}\sin(5t)\}$

12. Find the following inverse Laplace transforms.

(i) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}$

(ii) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+5}\right\}$

13. The matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

has eigenvectors

$$\xi_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Solve the initial value problem

$$x' = Ax, \qquad x(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

14. The matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

has eigenvectors

$$\xi_1 = \begin{bmatrix} 1 \\ 1/2 + i/2 \end{bmatrix} \qquad \text{and} \qquad \xi_2 = \begin{bmatrix} 1 \\ 1/2 - i/2 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = -2 + i$ and $\lambda_2 = -2 - i$. Find the solution to the initial value problem

$$x' = Ax, \qquad x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$