Math 285 Homework 4 Version A

1. Find the general solution to the Bernoulli differential equation

$$\dot{x} + 2x = tx^3.$$

Solution. Since $n = 3$ substitute $u = y^{1-n} = x^{-2}$ to obtain

$$\dot{u} = -2x^{-3}\dot{x} = -2x^{-3}(-2x + tx^3) = 4x^{-2} - 2t = 4u - 2t.$$

This is a linear equation with integrating factor $I = e^{-4t}$. Thus

$$\frac{d}{dt}(ue^{-4t}) = -2te^{-4t}.$$

Integration by parts obtains

$$\begin{aligned} ue^{-4t} &= -2\int te^{-4t}dt = \frac{1}{2}\int t\,de^{-4t} \\ &= \frac{1}{2}\Big\{te^{-4t} - \int e^{-4t}dt\Big\} = \frac{1}{2}\Big\{te^{-4t} + \frac{1}{4}e^{-4t}\Big\} + C \end{aligned}$$

Therefore

$$u = \frac{t}{2} + \frac{1}{8} + Ce^{4t} = \frac{4t + 1 + Ce^{4t}}{8}$$

and consequently the general solution is

$$y = \frac{1}{\sqrt{u}} = \frac{2\sqrt{2}}{\sqrt{4t + 1 + Ce^{4t}}}.$$

2. Find the unique solution to the initial value problem

$$\dot{x} + 2x = tx^3, \qquad x(t_0) = x_0$$

when $t_0 = 0$ and $x_0 = 4$.

Solution. Solve for the constant in the general solution. Since

$$y(0) = \frac{2\sqrt{2}}{\sqrt{4 \cdot 0 + 1 + Ce^{4 \cdot 0}}} = \frac{2\sqrt{2}}{\sqrt{1 + C}} = 4.$$

Therefore

$$1 + C = \frac{1}{2}$$

and consequently the unique solution is

$$y = \frac{2\sqrt{2}}{\sqrt{4t + 1 - (1/2)e^{4t}}} = \frac{4}{\sqrt{8t + 2 - e^{4t}}}.$$

3. Show that the solution found above blows up sometime between $t = 0.4$ and 0.5.

Solution. The solution blows up when the function $q(t) = 8t + 2 - e^{4t}$ that is inside the square root in the denominator is zero. Since q is continuous and

$$q(0.4) = 8 \cdot 0.4 + 2 - e^{4 \cdot 0.4} \approx 0.2470 > 0$$

and

$$q(0.5) = 8 \cdot 0.5 + 2 - e^{4 \cdot 0.5} \approx -1.3891 < 0,$$

then the intermediate value theorem implies there is some point c between 0.4 and 0.5 where $q(c) = 0$. Consequently, the solution y blows up at $t = c$.

4. Solve $8t + 2 = e^{4t}$ to 4 significant digits to find the approximate time of blow up.

Solution. There are many ways to find the point c such that $q(c) = 0$. A good method is Newton's method

$$c_{i+1} = \phi(c_i) \qquad \text{where} \qquad \phi(t) = t + \frac{q(t)}{q'(t)} = t + \frac{2 + 8t - e^{4t}}{8 - 4e^{4t}}.$$

with an initial guess of $c_0 = 0.45$. While this calculation may be done with a hand calculator, the Matlab script

```
phi=@(t)t-(2+8*t-exp(4*t))/(8-4*exp(4*t));
c(1)=0.45;
for i=1:5
    c(i+1)=phi(c(i));
end;
c
```

can also be used. The output is

```
c =

   0.45000   0.42224   0.41961   0.41959   0.41959   0.41959
```

which shows that $c \approx 0.4196$ to 4 significant digits.

5. Use Euler's method

$$x_{k+1} = x_k + hf(x_k, t_k) \qquad \text{for} \qquad k = 0, 1, \ldots, n-1$$

with $h = 0.4/n$ and $t_k = kh$ to approximate $x(0.4)$. Compute the error

$$\mathcal{E}_{\text{Euler}} = \big|x_n - x(0.4)\big|$$

for values of n equal to 5, 10, 20, 40 and 80.

Solution. This calculation can be running the Matlab script developed in class once for each value of n. The following script allows all the calculations to be done at once.

```
y=@(t)4/sqrt(8*t+2-exp(4*t));
f=@(x,t)t*x^3-2*x;
t(1)=0;
x(1)=4;
T=0.4;
i=1;
for n=[5 10 20 40 80]
    h=T/n;
    for k=1:n
        x(k+1)=x(k)+h*f(x(k),t(k));
        t(k+1)=t(1)+h*k;
    end;
    N(i)=n;
    X(i)=x(n+1);
    E(i)=abs(x(n+1)-y(T));
    i=i+1;
end;
[N
X
E]
```

The output is

```
ans =

    5.0000   10.0000   20.0000   40.0000   80.0000
    3.1534    3.9792    4.8328    5.6884    6.4641
    4.8956    4.0698    3.2162    2.3606    1.5849
```

where the first row gives the values for n, the second gives the approximation x_n and the third row gives the corresponding error $\mathcal{E}_{\text{Euler}}$. Compared to the true solution

$$x(0.4) = \frac{4}{\sqrt{8 \cdot 0.4 + 2 - e^{4 \cdot 0.4}}} \approx 8.0490$$

the errors are all relatively large. In particular, none of the numerical approximations are correct even in the first digit.

6. Use the Runge-Kutta RK4 method

$$\begin{aligned} f_1 &= f(x_k, t_k) \\ f_2 &= f(x_k + \tfrac{1}{2}hf_1, t_k + \tfrac{1}{2}h) \\ f_3 &= f(x_k + \tfrac{1}{2}hf_2, t_k + \tfrac{1}{2}h) \\ f_4 &= f(x_k + hf_3, t_k + h) \\ x_{k+1} &= x_k + \tfrac{1}{6}h(f_1 + 2f_2 + 2f_3 + f_4) \end{aligned}$$

to approximate $x(0.4)$. Compute the error

$$\mathcal{E}_{\text{RK4}} = \big|x_n - x(0.4)\big|$$

for values of n equal to 5, 10, 20, 40 and 80.

Solution. After replacing Euler's method by the RK4 method in the previous Matlab script we obtain

```
y=@(t)4/sqrt(8*t+2-exp(4*t));
f=@(x,t)t*x^3-2*x;
t(1)=0;
x(1)=4;
T=0.4;
i=1;
for n=[5 10 20 40 80]
    h=T/n;
    for k=1:n
        f1=f(x(k),t(k));
        f2=f(x(k)+0.5*h*f1,t(k)+0.5*h);
        f3=f(x(k)+0.5*h*f2,t(k)+0.5*h);
        f4=f(x(k)+h*f3,t(k)+h);
        x(k+1)=x(k)+(h/6)*(f1+2*f2+2*f3+f4);
        t(k+1)=t(1)+h*k;
    end;
    N(i)=n;
    X(i)=x(n+1);
    E(i)=abs(x(n+1)-y(T));
    i=i+1;
end;
[N
X
E]
```

with output

```
ans =

   5.0000e+00   1.0000e+01   2.0000e+01   4.0000e+01   8.0000e+01
   7.7112e+00   7.9816e+00   8.0415e+00   8.0484e+00   8.0489e+00
   3.3778e-01   6.7372e-02   7.5093e-03   5.2910e-04   2.9067e-05
```

The error is much smaller. In particular, all approximations are correct to at least one significant digit and when $n = 80$ the approximation is good to 5 significant digits.

7. Comment on the accuracy of the above numerical approximations.

Solution. Expanding upon the commentary above we recall that the errors in the Euler method were much larger for the same values of n. However, the steps for the RK4 method require 4 evaluations of $f(x,t)$ and are, therefore, approximately 4 times more expensive than the Euler method. Adjusted for similar computational work, it is more appropriate to compare

$$\mathcal{E}_{\text{Euler}} = 1.5849 \qquad \text{for} \qquad n = 80$$

to

$$\mathcal{E}_{\text{RK4}} = 0.0075093 \qquad \text{for} \qquad n = 20.$$

Either method requires 80 evaluations of $f(x,t)$. However, for this amount of computational work, the RK4 method is more than 200 times more accurate than the Euler method. As greater accuracy is required, the advantages of RK4 over Euler's method become even more pronounced.

8. [Extra Credit 3] Approximate $x(0.2)$ where $x(t)$ is the unique solution to

$$\dot{x} - 4\sin x = tx^3, \qquad x(0) = 4.$$

This solution also blows up at some time $t > 0$. Numerically approximate the time of blow up to 4 significant digits.

Solution. Since $|\sin x| \le 1$ we know that any solution satisfies

$$\dot{x} \ge tx^3 - 4.$$

Suppose there is $t_1 \in (0.25, 0.5)$ such that $x_1 = x(t_1) \ge 201$. Then

$$tx \ge 50.25 > 4 \qquad \text{for} \qquad t = t_1.$$

By continuity there is $\delta > 0$ such that

$$tx^3 - 4 = tx^3 - tx + tx - 4 > tx^3 - tx = tx(x^2 - 1) \qquad \text{for} \qquad t \in [t_1, t_1 + \delta].$$

Integrating for $t \in [t_1, t_1 + \delta]$ we have

$$\int_{x_1}^{x} \frac{dx}{x(x-1)(x+1)} \ge \int_{t_1}^{t} t\,dt.$$

Since

$$\frac{1}{x(x-1)(x+1)} = \frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{1}{x}$$

then

$$\frac{1}{2}\log\frac{x+1}{x_1+1} + \frac{1}{2}\log\frac{x-1}{x_1-1} - \log\frac{x}{x_1} \ge \frac{1}{2}(t^2 - t_1^2)$$

and consequently

$$\frac{1 - x^{-2}}{1 - x_1^{-2}} \ge \exp(t^2 - t_1^2) \qquad \text{for} \qquad t \in [t_1, t_1 + \delta].$$

Since this inequality implies x is strictly increasing for $t \ge t_1$, then the inequality must hold until the time of blowup. Let t_* be the time of blowup. Thus,

$$x^2 \ge \frac{1}{1 - (1 - x_1^{-2})\exp(t^2 - t_1^2)} \qquad \text{for} \qquad t \in [t_1, t_*).$$

Since the denominator on the right hand side goes to zero when

$$(1 - x_1^{-2})\exp(t^2 - t_1^2) = 1$$

this implies

$$t_* - t_1 \le \frac{-\log(1 - x_1^{-2})}{t_* + t_1} < \frac{0.000025}{0.5} = 0.00005.$$

Consequently $|t_* - t_1| < 0.00005$. In particular $t_* \approx t_1$ to 4 significant digits. It remains to numerically find a point t_1 such that $x \geq 201$ when $t = t_1$. The Matlab script

```
f=@(x,t)t*x^3+4*sin(x);
t(1)=0;
x(1)=4;
T=1;
i=1;
n=80000;
h=T/n;
for k=1:n
    f1=f(x(k),t(k));
    f2=f(x(k)+0.5*h*f1,t(k)+0.5*h);
    f3=f(x(k)+0.5*h*f2,t(k)+0.5*h);
    f4=f(x(k)+h*f3,t(k)+h);
    x(k+1)=x(k)+(h/6)*(f1+2*f2+2*f3+f4);
    t(k+1)=t(1)+h*k;
    if (x(k+1)>=201)
        [t(k+1),x(k+1)]
        exit;
    end;
end;
```

produces the output

```
ans =

     0.28590   228.70080
```

which shows that $x(0.2859) \approx 228.7008$. Since $0.2859 \in (0.25, 0.5)$, the point of blowup

$$t_* \approx 0.2859$$

to 4 significant digits.