Math 285 Sample Final Version A

Instructions. This exam consists of two parts. The first part will be graded with partial credit based on how well you explain your answer; the second part is similar to the computer graded homework with limited partial credit. On the second part, only the answers you write inside the boxes will be graded.

1. What is a differential equation?

2. What is a Bernoulli differential equation and what substitution helps solve it?

3. What does it mean for the functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ to be linearly dependent?

4. State the definition of the Laplace transform.

5. Find the solution $y(x)$ to the differential equation

$$x\frac{dy}{dx} - 4y = x^6 e^x \qquad \text{such that} \qquad y(1) = 2.$$

6. Solve the differential equation

$$\frac{dy}{dx} = e^{5x+6y} \qquad \text{such that} \qquad y(0) = 0$$

by separation of variables.

$y(x) =$ $\boxed{}$

7. Determine whether the differential equation

$$(2xy^2 - 6)\,dx + (2x^2y + 3)\,dy = 0.$$

is exact. If it is exact, solve it; if it is not exact, write *not* in the box.

$\boxed{}$

8. A tank contains 90 liters of fluid in which 20 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 3 L/min; the well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of grams of salt in the tank at time t.

$A(T) =$ $\boxed{}$ grams.

9. Determine whether the set of functions

$$f_1(x) = x, \qquad f_2(x) = x^2 \qquad \text{and} \qquad f_3(x) = 4x - 7x^2$$

is linearly independent on the interval $(-\infty, \infty)$.

(A) linearly dependent

(B) linearly independent

10. Find the general solution of the higher-order differential equation

$$y''' - 9y'' + 15y' + 25y = 0.$$

The general solution is

$y(t) =$ $\boxed{}$

11. Find the general solution to the differential equation

$$y'' + 4y = 2\sin 2x$$

using undetermined coefficients. The general solution is

$y(t) =$

12. Solve the differential equation

$$x^2y'' + 7xy' + 9y = 0 \qquad \text{for} \qquad x > 0.$$

The general solution for $x > 0$ is

$y(x) =$

13. Use the table of Laplace transforms attached to find $\mathcal{L}\{f(t)\}(s)$ where

$$f(t) = 4t^2 - 5\sin 2t.$$

$\mathcal{L}\{f(t)\}(s) =$

14. Use the table of Laplace transforms attached to find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^5}\right\}(t).$$

Write your answer as a function of t.

15. Use the Laplace transform to solve the initial-value problem

$$y' + 4y = e^{6t} \qquad \text{such that} \qquad y(0) = 2.$$

The answer is

$y(t) =$ $\boxed{}$

16. Find the Laplace transform

$$F(s) = \mathcal{L}\{t(e^t + e^{3t})^2\}(s).$$

The Laplace transform is

$F(s) =$ $\boxed{}$

17. Find the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}(t).$$

The inverse Laplace transform is

$f(t) =$ $\boxed{}$

18. Write the function

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t < 1 \\ t^2 & \text{for } t \ge 1. \end{cases}$$

in terms of unit step functions. Find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}(s)$.

$F(s) =$ $\boxed{}$

19. Use the convolution theorem to evaluate the Laplace transform

$$F(s) = \mathcal{L}\left\{\int_0^t \tau e^{t-\tau}\, d\tau\right\}(s).$$

The Laplace transform is

$F(s) =$ $\boxed{}$

20. Use the Laplace transform to solve the initial-value problem

$$y' - 4y = \delta(t-5) \qquad \text{such that} \qquad y(0) = 0.$$

The solution is

$y(t) =$ $\boxed{}$

21. Suppose

$$A = \begin{bmatrix} 1 & 6 \\ 7 & 11 \\ 10 & 12 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -6 & 8 & -3 \\ 1 & -3 & 2 \end{bmatrix}$$

Find the products

$AB =$ $\boxed{}$ $\qquad BA =$ $\boxed{}$

22. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = 4, \qquad \lambda_2 = -2 \qquad \text{and} \qquad \lambda_3 = -1$$

and corresponding eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \qquad \text{and} \qquad K_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Find the general solution of the matrix differential equation $X' = AX$.

$X(t) =$ $\boxed{}$

TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. t	$\dfrac{1}{s^2}$
3. t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5. $t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6. t^{α}	$\dfrac{\Gamma(\alpha + 1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\dfrac{k}{s^2 + k^2}$
8. $\cos kt$	$\dfrac{s}{s^2 + k^2}$
9. $\sin^2 kt$	$\dfrac{2k^2}{s(s^2 + 4k^2)}$
10. $\cos^2 kt$	$\dfrac{s^2 + 2k^2}{s(s^2 + 4k^2)}$
11. e^{at}	$\dfrac{1}{s - a}$
12. $\sinh kt$	$\dfrac{k}{s^2 - k^2}$
13. $\cosh kt$	$\dfrac{s}{s^2 - k^2}$
14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2 - 4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2 - 2k^2}{s(s^2 - 4k^2)}$
16. te^{at}	$\dfrac{1}{(s - a)^2}$
17. $t^n e^{at}$	$\dfrac{n!}{(s - a)^{n+1}}$, n a positive integer
18. $e^{at} \sin kt$	$\dfrac{k}{(s - a)^2 + k^2}$
19. $e^{at} \cos kt$	$\dfrac{s - a}{(s - a)^2 + k^2}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
20. $e^{at} \sinh kt$	$\dfrac{k}{(s - a)^2 - k^2}$
21. $e^{at} \cosh kt$	$\dfrac{s - a}{(s - a)^2 - k^2}$
22. $t \sin kt$	$\dfrac{2ks}{(s^2 + k^2)^2}$
23. $t \cos kt$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$
24. $\sin kt + kt \cos kt$	$\dfrac{2ks^2}{(s^2 + k^2)^2}$
25. $\sin kt - kt \cos kt$	$\dfrac{2k^3}{(s^2 + k^2)^2}$
26. $t \sinh kt$	$\dfrac{2ks}{(s^2 - k^2)^2}$
27. $t \cosh kt$	$\dfrac{s^2 + k^2}{(s^2 - k^2)^2}$
28. $\dfrac{e^{at} - e^{bt}}{a - b}$	$\dfrac{1}{(s - a)(s - b)}$
29. $\dfrac{ae^{at} - be^{bt}}{a - b}$	$\dfrac{s}{(s - a)(s - b)}$
30. $1 - \cos kt$	$\dfrac{k^2}{s(s^2 + k^2)}$
31. $kt - \sin kt$	$\dfrac{k^3}{s^2(s^2 + k^2)}$
32. $\dfrac{a \sin bt - b \sin at}{ab(a^2 - b^2)}$	$\dfrac{1}{(s^2 + a^2)(s^2 + b^2)}$
33. $\dfrac{\cos bt - \cos at}{a^2 - b^2}$	$\dfrac{s}{(s^2 + a^2)(s^2 + b^2)}$
34. $\sin kt \sinh kt$	$\dfrac{2k^2 s}{s^4 + 4k^4}$
35. $\sin kt \cosh kt$	$\dfrac{k(s^2 + 2k^2)}{s^4 + 4k^4}$
36. $\cos kt \sinh kt$	$\dfrac{k(s^2 - 2k^2)}{s^4 + 4k^4}$
37. $\cos kt \cosh kt$	$\dfrac{s^3}{s^4 + 4k^4}$
38. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2 + k^2}}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
39. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s-a}{s-b}$
40. $\dfrac{2(1 - \cos kt)}{t}$	$\ln \dfrac{s^2 + k^2}{s^2}$
41. $\dfrac{2(1 - \cosh kt)}{t}$	$\ln \dfrac{s^2 - k^2}{s^2}$
42. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
43. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2}\arctan \dfrac{a+b}{s} + \dfrac{1}{2}\arctan \dfrac{a-b}{s}$
44. $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
45. $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
46. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
47. $2\sqrt{\dfrac{t}{\pi}} e^{-a^2/4t} - a \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
48. $e^{ab} e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
49. $-e^{ab} e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$
50. $e^{at} f(t)$	$F(s-a)$
51. $\mathscr{U}(t-a)$	$\dfrac{e^{-as}}{s}$
52. $f(t-a)\mathscr{U}(t-a)$	$e^{-as}F(s)$
53. $g(t)\mathscr{U}(t-a)$	$e^{-as}\mathscr{L}\{g(t+a)\}$
54. $f^{(n)}(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
55. $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
56. $\displaystyle\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$
57. $\delta(t)$	1
58. $\delta(t - t_0)$	e^{-st_0}