Math 285 Exam 1 Version B — *Key*

Instructions. This exam consists of two parts. The first part will be graded with partial credit based on how well you explain your answer; the second part is similar to the computer graded homework with limited partial credit. On the second part, only the answers you write inside the boxes will be graded.

1. What is a differential equation?

an equation containing the derivatives of one or more functions (or dependent variables), with respect to one or more independent variables, is said to be a differential equation.

2. What is a Bernoulli differential equation and what substitution helps solve it?

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)\,y^n,$$

where n is any real number, is called Bernoulli's equation. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces Bernoulli's equation to a linear equation.

3. What does it mean for the functions $f_1(x), f_2(x), \ldots, f_n(x)$ to be linearly dependent?

a set of functions $f_1(x), \ldots, f_n(x)$ is said to be linearly dependent on an interval I if there exist constants $c_1, \ldots, c_n$ not all zero such that

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval.

4. State the definition of the Laplace transform.

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

is said to be the Laplace transform of f, provided that the integral converges.

5. Find the solution $y(x)$ to the differential equation

$$x\frac{dy}{dx} - 3y = x^5 e^{-x} \qquad \text{such that} \qquad y(1) = 4.$$

Standard form:

$$y' - \frac{3}{x}y = x^4 e^{-x}$$

integrating factor:

$$\mu = e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$$

Thus

$$(yx^{-3})' = xe^{-x}$$

Integrate

$$yx^{-3} = \int xe^{-x}dx = -\int x\,de^{-x}$$

$$= -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + c$$

$$= -(x+1)e^{-x} + c$$

Thus

$$y = -x^3(x+1)e^{-x} + cx^3$$

Solve for c

$$y(1) = -2e^{-1} + c = 4, \qquad c = 4 + 2e^{-1}$$

Unique solution

$$y(x) = -x^3(x+1)e^{-x} + (4+2e^{-1})x^3.$$

6. Determine whether the differential equation

$$(3x^2y + 2x)\,dx + (x^3 - 3)\,dy = 0.$$

is exact. If it is exact, solve it; if it is not exact, write *not* in the box.

$$\boxed{x^3y + x^2 - 3y = C \quad \text{or} \quad y = \frac{C - x^2}{x^3 - 3}}$$

7. A tank contains 90 liters of fluid in which 20 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 3 L/min; the well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of grams of salt in the tank at time t.

$A(t) = \boxed{90 - 70e^{-\frac{1}{30}t}}$ grams.

8. Find the general solution to the differential equation

$$y'' - 4y = 2e^{2x}.$$

using undetermined coefficients. The general solution is

$y(x) = \boxed{c_1e^{2x} + c_2e^{-2x} + \frac{1}{2}xe^{2x}}$

9. Solve the differential equation

$$x^2y'' - 4xy' + 6y = 0 \qquad \text{for} \qquad x > 0.$$

The general solution for $x > 0$ is

$y(x) = \boxed{c_1x^3 + c_2x^2}$

10. Use the table of Laplace transforms attached to find $\mathcal{L}\{f(t)\}(s)$ where

$$f(t) = 5t^3 - \cos 2t.$$

$$\mathcal{L}\{f(t)\}(s) = \boxed{\frac{30}{s^4} - \frac{s}{s^2+4}}$$

11. Use the table of Laplace transforms attached to find the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{s^4}\right\}(t).$$

Write your answer as a function of t.

$$\boxed{t + t^2 + \frac{t^3}{6}}$$

12. Use the Laplace transform to solve the initial-value problem

$$y' + 4y = e^{6t} \qquad \text{such that} \qquad y(0) = 2.$$

The answer is

$$y(t) = \boxed{\frac{1}{10} e^{6t} + \frac{19}{10} e^{-4t}}$$

13. Find the Laplace transform

$$F(s) = \mathcal{L}\{t(e^{2t} - 3e^{-t})^2\}(s).$$

The Laplace transform is

$$F(s) = \boxed{\frac{1}{(s-4)^2} - \frac{6}{(s-1)^2} + \frac{9}{(s+2)^2}}$$

14. Find the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+2)}\right\}(t).$$

The inverse Laplace transform is

$$f(t) = \boxed{\frac{1}{2} \mathcal{U}(t-1)(1 - e^{2-2t})}$$