— Key —

Math 285 Final Supplement Study Questions Version A

1. Use the convolution theorem to evaluate the Laplace transform

$$F(s) = \mathcal{L}\Big\{ \int_0^t \tau e^{t-\tau} d\tau \Big\}(s).$$

The Laplace transform is

$$F(s) = \boxed{\frac{1}{s^2(s-1)}}$$

2. Suppose

$$A = \begin{bmatrix} 1 & 6 \\ 7 & 11 \\ 10 & 12 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -6 & 8 & -3 \\ 1 & -3 & 2 \end{bmatrix}$$

Find the products

$$AB = \boxed{\begin{bmatrix} 0 & -10 & 9 \\ -31 & 23 & 1 \\ -48 & 44 & -6 \end{bmatrix}} \qquad BA = \boxed{\begin{bmatrix} 20 & 16 \\ 0 & -3 \end{bmatrix}}$$

3. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = 4, \qquad \lambda_2 = -2 \qquad \text{and} \qquad \lambda_3 = -1$$

and corresponding eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \qquad \text{and} \qquad K_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Find the general solution of the matrix differential equation $X' = AX$.

$$X(t) = \boxed{c_1 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}}$$

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4. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = 1 + 2i \qquad \text{and} \qquad \lambda_2 = 1 - 2i$$

and corresponding eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad \text{and} \qquad K_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Find the unique solution of the matrix differential equation

$$X' = AX, \qquad X(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

$$X(t) = \boxed{\begin{bmatrix} 3\cos 2t + 4\sin 2t \\ 4\cos 2t - 3\sin 2t \end{bmatrix} e^t}$$

5. Consider the matrix

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$$

with an eigenvalue

$$\lambda_1 = 3 \quad \text{of multiplicity} \quad 2$$

and corresponding eigenvector

$$K_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Note that this is the case where there is no second eigenvector. Find the general solution of the matrix differential equation $X' = AX$.

$$X(t) = \boxed{c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t \right) e^{3t}}$$

#1 $$\int_0^t \tau e^{t-\tau}\,d\tau = (f*g)(t)$$

where $f(t)=t$ and $g(t)=e^t$

$$\mathcal{L}\left\{\int_0^t \tau e^{t-\tau}\,d\tau\right\}(s) = \mathcal{L}\{(f*g)(t)\}(s)$$

$$= \mathcal{L}\{f(t)\}(s)\,\mathcal{L}\{g(t)\}(s)$$

$$= \mathcal{L}\{t\}(s)\,\mathcal{L}\{e^t\}(s)$$

$$= \frac{1}{s^2}\,\frac{1}{s-1} = \frac{1}{s^2(s-1)}$$

#4 Complex eigenvalues and eigenvectors

$$\lambda_1 = 1+2i \qquad K_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Thus

$$X_1(t) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t \right) e^t$$

$$X_2(t) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 2t \right) e^t$$

General solution

$$X(t) = c_1 X_1 + c_2 X_2$$

$$= c_1 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t \right) e^t$$

$$+ c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 2t \right) e^t$$

Unique solution such that $X(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$X(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$c_1 = 3 \quad \text{and} \quad c_2 = 4$$

Therefore

$$X(t) = 3 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 2t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t \right) e^t$$

$$+ 4 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 2t \right) e^t$$

$$= \left(\begin{bmatrix} 3\cos 2t \\ -3\sin 2t \end{bmatrix} + \begin{bmatrix} 4\sin 2t \\ 4\cos 2t \end{bmatrix} \right) e^t$$

$$= \begin{bmatrix} 3\cos 2t + 4\sin 2t \\ 4\cos 2t - 3\sin 2t \end{bmatrix} e^t .$$

5 Repeated eigenvalues and missing eigenvectors

$$X_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

$$X_2(t) = (B + Kt)e^{3t}$$

$$X_2' = Ke^{3t} + 3(B+Kt)e^{3t}$$

$$= (K+3B)e^{3t} + 3Kte^{3t}$$

$$MX_2 = (MB + MKt)e^{3t}$$

$MK = 3K$ implies $K = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$MB = K + 3B$, $(M-3)B = K$

Therefore

$$\left(\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$2b_1 - b_2 = 1$, $b_2 = 2b_1 - 1$, $B = \begin{bmatrix} b_1 \\ 2b_1 - 1 \end{bmatrix}$

Taking $b_1 = 0$ implies $B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$X_2(t) = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t \right) e^{3t}$$