Linear first order differential equations.

DEFINITION 2.3.1 Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{1}$$

is said to be a linear equation in the variable y.

$$\frac{dy}{dx} + \underbrace{\frac{a_0(x)}{a_1(x)}}_{P(x)} y = \underbrace{\frac{g(x)}{a_1(x)}}_{f(x)}$$

Try to solve

$$\frac{dy}{dx} + P(x)y = f(x)$$

What if $P(x) = 0$? Then it's just antidifferentiation.

$$\frac{dy}{dx} = f(x), \qquad y = \int f(x)\,dx$$

What if $f(x) = 0$ and $P(x) = 3$. Then separation of vbls

$$\frac{dy}{dx} + 3y = 0 \qquad \frac{dy}{dx} = -3y$$

$$\int \frac{dy}{y} = \int -3\,dx \qquad \ln y = -3x + C$$

$$y = e^{-3x+C} = e^{-3x}e^{C} = Ae^{-3x}$$

where $A = e^{C}$

What if $f(x)=0$ and $p(x)\neq 0$,

$$y' + P(x)\,y = 0 \qquad \frac{dy}{dx} = -P(x)\,y$$

$$\int \frac{dy}{y} = \int -P(x)\,dx \qquad \ln y = -\int P(x)\,dx$$

$$y = e^{-\int P(x)dx}$$

Now suppose $f(x)\neq 0$ and $P(x)\neq 0$.

$$y' + P(x)y = f(x) \qquad y' = f(x) - P(x)y$$

Idea use the solution of the simpler equation to simplify the more complicated equation...

$$y(x) = u(x)e^{-\int P(x)dx}$$

$\mu(x) = e^{\int p(x)dx}$

$$y(x) = \frac{u(x)}{\mu(x)}$$

taking $u(x) = y(x)e^{\int P(x)dx}$ then we see this any solution can be written this way...

$$u'(x) = \frac{d}{dx}\left(y(x)e^{\int P(x)dx}\right) = y'e^{\int P(x)dx} + y\frac{d}{dx}e^{\int P(x)dx}$$

$$= y'e^{\int p(x)dx} + y\,e^{\int P(x)dx}p$$

$$= \left(f(x) - P(x)\cancel{y}\right)e^{\int p(x)dx} + y\,\cancel{e^{\int P(x)dx}}p = f(x)\,e^{\int p(x)dx}$$

$$\frac{d}{dx}e^{\int P(x)dx} = e^{\int P(x)dx}\frac{d}{dx}\int P(x)dx = e^{\int P(x)dx}p$$

Therefore

$$u'(x) = f(x)\,e^{\int p(x)dx} \qquad u(x) = \underbrace{\int \left(f(x)\,e^{\int p(x)dx}\right)dx}_{\text{crazy because 2 dx's appear}}$$

Fix notation

$$\mu(x) = e^{\int p(x)dx}$$

$$u(x) = \int f(x)\,\mu(x)\,dx$$

only one dx per equation...

Example 2 (book page 56 in 10^{th} Edition)

$$\frac{dy}{dx} - 3y = 6$$

for μ we can set the constant to anything that is convenient.

$P(x) = -3 \qquad f(x) = 6$

$$\mu(x) = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x + C_2}$$

Why?

$$\mu(x) = e^{-3x}$$

$$u(x) = \int f(x)\mu(x)\,dx = \int 6e^{-3x}\,dx = \frac{6}{-3}e^{-3x} + C$$

$$y(x) = u(x)e^{-\int P(x)dx} = \left(\frac{6}{-3}e^{-3x} + C\right)e^{-\int P(x)dx}$$

$$= \frac{\frac{6}{-3}e^{-3x} + C}{e^{\int P(x)dx}} = \frac{-2e^{-3x} + C}{\mu(x)} = \frac{-2e^{-3x+C_2} + C}{e^{-3x+C_2}}$$

If we left the C_2 in they would have cancelled in the first term and combined to make a different constant in the second term...

Therefore

$$y(x) = \frac{-2e^{-3x} + C}{e^{-3x}} = -2 + \frac{C}{e^{-3x}} = -2 + Ce^{3x}$$

Check solution:

$$\frac{dy}{dx} - 3y = 6 \qquad y = -2 + Ce^{3x}$$

$$y' = 3Ce^{3x}$$
$$-3y = 6 - 3Ce^{3x}$$
$$y' - 3y = 6 \quad \boxed{\checkmark}$$

9. $x\frac{dy}{dx} - y = x^2 \sin x$ assume $x > 0$

$$y' - \frac{1}{x}y = \underbrace{x \sin x}_{f(x)}$$

$$P(x) = -\frac{1}{x}, \quad \mu(x) = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$u(x) = \int (x \sin x)\frac{1}{x}\,dx = -\cos x + C$$

$$y(x) = \frac{u(x)}{\mu(x)} = \frac{-\cos x + C}{\left(\frac{1}{x}\right)} = -x\cos x + Cx$$

Remark: solution seems to make all sense even when $x<0$ since the ln are gone...

Check: $x\frac{dy}{dx} - y = x^2 \sin x$

$$y = -x\cos x + Cx$$
$$y' = x\sin x - \cos x + C$$

$$xy' = x^2 \sin x - x\cos x + Cx$$
$$-y = x\cos x - Cx$$
$$xy' - y = x^2 \sin x$$

worked for both positive and negative x values...