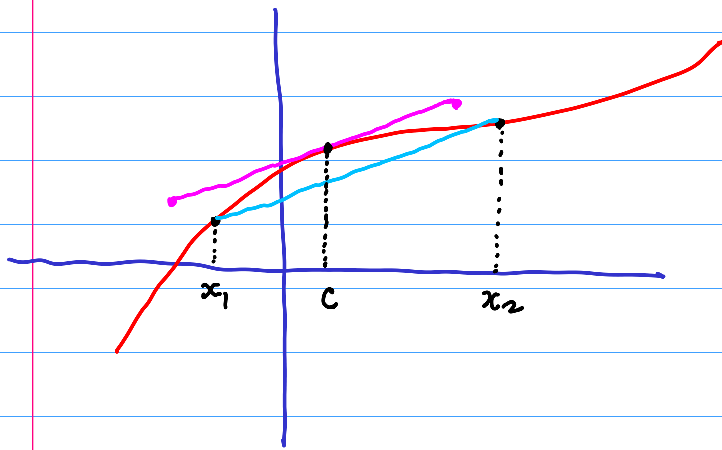
Mean Value Theorem



$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$$

or

$$f(x_2)-f(x_1) = f'(c)\left(x_2-x_1\right)$$

Corollary: If $u'(x)=0$ for all $x \in [a,b]$ then the function u is constant.

Let $x_1, x_2 \in [a,b]$. Then for some c between x_1 and x_2

$$u(x_2)-u(x_1) = \underbrace{u'(c)}_{=0}\left(x_2-x_1\right) = 0$$

so

$$u(x_2)-u(x_1) = 0 \qquad \text{or} \qquad u(x_1) = u(x_2)$$

Since this holds for any $x_1, x_2 \in [a,b]$, then u is constant.

First order ordinary differential equation

$$y' + P(x)\,y = f(x)$$

$$\mu(x) = e^{\int P(x)\,dx} \qquad \text{integrating factor ...}$$

Back to the case $f(x)=0$

$$\boxed{y' + P(x)\,y = 0} \qquad \text{homogenous equation}$$

the y is in every term, so the equation scales with changes in the scale of y and remains equal.

Now, by product rule

$$(y\mu)' = y'\mu + y\mu' = y'\mu + y\mu p = \mu(y' + yp) = 0$$

$$\mu'(x) = \frac{d}{dx} e^{\int p(x)\,dx} = \underbrace{e^{\int p(x)\,dx}}_{\mu} \frac{d}{dx}\int p(x)\,dx = \mu(x)\,p(x)$$

Thus, $(y\mu)' = 0$ so $y\mu$ is constant by the mean value theorem.

So $y\mu = C$ for some C.

$$y(x) = \frac{C}{\mu(x)} = \frac{C}{e^{\int p(x)\,dx}} = C e^{-\int p(x)\,dx}$$

all solutions are of this form

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25. $\dfrac{dy}{dx} = x + 5y, \quad y(0) = 3$

$$\underbrace{y' - 5y}_{p(x) = -5} = \underbrace{x}_{f(x) = x}$$

$$\mu(x) = e^{\int p(x)\,dx} = e^{-\int 5\,dx} = e^{-5x}$$

Note μ is actually the solution to the differential equation $\mu' = -5\mu$

$$(y\mu)' = y'\mu + y\mu' = y'\mu + y(-5\mu) = y'\mu - 5y\mu = \mu(y' - 5y)$$

left side of differential equation hiding inside.

$$(y\mu)' = \mu(y' - 5y) = \mu x$$

$$y\mu = \int \mu x\, dx = \int e^{-5x} x\, dx$$

Using integration by parts

$$\int e^{-5x} x\, dx = \int \underset{u}{x}\, \underset{dv}{d\left(\frac{1}{-5} e^{-5x}\right)} = \underset{u}{x}\, \underset{v}{\frac{1}{-5} e^{-5x}} - \int \underset{v}{\frac{1}{-5} e^{-5x}}\, \underset{du}{dx}$$

$$= \frac{-x}{5} e^{-5x} - \frac{1}{5^2} e^{-5x} + C = -\frac{1}{5}\left(x + \frac{1}{5}\right) e^{-5x} + C$$

Therefore

$$y = \frac{1}{\mu}\left(-\frac{1}{5}\left(x + \frac{1}{5}\right) e^{-5x} + C\right) = -\frac{1}{5}\left(x + \frac{1}{5}\right) + C e^{5x}$$

Check answer. plug $y = -\frac{1}{5}\left(x + \frac{1}{5}\right) + C e^{5x}$ into

into. $\frac{dy}{dx} = x + 5y,$

Now solve for C so that $y(0) = 3$

$$\begin{array}{r} 25 \\ \times 3 \\ \hline 75 \end{array}$$

$$y(0) = -\frac{1}{5}\left(0 + \frac{1}{5}\right) + C e^{5 \cdot 0} = -\frac{1}{5^2} + C = 3$$

So $C = 3 + \frac{1}{5^2} = \frac{76}{25}$

Final answer:

$$y = -\frac{1}{5}\left(x + \frac{1}{5}\right) + \frac{76}{25} e^{5x}$$

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$$y' + 4xy = x^3 e^{x^2} \qquad y(0) = -1$$

$$\mu = e^{\int 4x\,dx} = e^{2x^2}$$

$$\left(y e^{2x^2}\right)' = x^3 e^{x^2} e^{2x^2} = x^3 e^{3x^2}$$

$$y e^{2x^2} = \int x^3 e^{3x^2}\,dx = \int \frac{1}{3}\frac{1}{6} \underbrace{3x^2}_{u} e^{3x^2} \underbrace{6x\,dx}_{du}$$

$u = 3x^2$

$du = 6x\,dx$

$$= \frac{1}{18}\int u e^u\,du = \frac{1}{18}\int u\,de^u = \frac{1}{18}\left(ue^u - \int e^u\,du\right)$$

$$= \frac{1}{18}\left(ue^u - e^u\right) + C = \frac{1}{18}\left(3x^2 - 1\right)e^{3x^2} + C$$