#32 from section 2.3

$$y' + 4xy = x^3 e^{x^2} \qquad y(0) = -1$$

$$\mu = e^{\int 4x\,dx} = e^{2x^2}$$

$$\left(y e^{2x^2}\right)' = x^3 e^{x^2} e^{2x^2} = x^3 e^{3x^2}$$

$$y e^{2x^2} = \int x^3 e^{3x^2}\,dx = \int \frac{1}{3}\frac{1}{6} \underbrace{3x^2}_{u} e^{3x^2} \underbrace{6x\,dx}_{du}$$

$u = 3x^2$

$du = 6x\,dx$

$$= \frac{1}{18}\int u e^u\,du = \frac{1}{18}\int u\,de^u = \frac{1}{18}\left(ue^u - \int e^u\,du\right)$$

$$= \frac{1}{18}\left(ue^u - e^u\right) + C = \frac{1}{18}\left(3x^2 - 1\right)e^{3x^2} + C$$

Thus

$$y = \frac{1}{18}(3x^2 - 1)e^{x^2} + Ce^{-2x^2}$$

$$y(0) = \frac{1}{18}(3\cdot 0^2 - 1)e^{0^2} + Ce^{-2\cdot 0^2} = -\frac{1}{18} + C = -1$$

$$C = \frac{1}{18} - 1 = \frac{-17}{18}$$

Solution:

$$y = \frac{1}{18}(3x^2 - 1)e^{x^2} - \frac{17}{18}e^{-2x^2}$$

In general we are solving

$$y' + P(x)y = f(x)$$

$$\mu = e^{\int P(x)dx} \quad \text{since} \quad \mu' = \mu P(x)$$

mult. by integrating factor

$$\underbrace{y'\mu + P(x)y\mu}_{(y\mu)'} = f(x)\mu$$

$$(y\mu)' = f(x)\mu$$

Then

$$y\mu = \int f(x)\mu(x)\,dx$$

$$y = \frac{1}{\mu}\int f(x)\mu(x)\,dx$$

Exact equations:

When doing separation of variables we get an implicit solution of the form $f(x,y)=c$.

Find an ODE with $f(x,y)=c$ as solution by differentiating

$$\frac{d}{dx}f(x,y) = \frac{d}{dx}c$$

$$\frac{d}{dx}f(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\underbrace{\frac{\partial f}{\partial x}}_{M} + \underbrace{\frac{\partial f}{\partial y}}_{N}\frac{dy}{dx} = 0$$

these are known if we started with f.

$$M + N\frac{dy}{dx} = 0$$

or

$$M\,dx + N\,dy = 0$$

Idea... work backwards... to find f.

$$2xy\,dx + (x^2-1)\,dy = 0$$

Look for the function f directly (rather than separating vbls)

$$\frac{\partial f}{\partial x} = 2xy \qquad f(x,y) = x^2y + g(y)$$

treat as a constant

because partial derivative in x means holding y constant

$$\frac{\partial f}{\partial y} = \boxed{(x^2-1)} \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2y + g(y)\right) = \boxed{x^2 + g'(y)}$$

Thus $\quad \cancel{x^2}-1 = \cancel{x^2} + g'(y)$

$$g'(y) = -1 \qquad g(y) = -y + c_1$$

and $\quad f(x,y) = x^2y - y + c_1$

Solutions are $f(x,y) = c$ or

$$x^2y - y + c_1 = c$$

combine the constants

Solutions...

$$x^2y - y = C$$

So it's enough to write $f(x,y) = x^2y - y$ so that

$f(x,y) = c$ are the implicit solutions.