Try to work this one backwards..

$$xy\,dx + (2x^2+3y^2-20)\,dy = 0$$

$$\frac{\partial f}{\partial x} = xy \qquad\qquad f(x,y) = \frac{1}{2}x^2y + g(y)$$

treat y as a constant

$$\frac{\partial f}{\partial y} = \frac{1}{2}x^2 + g'(y) = (2x^2+3y^2-20)$$

solve for g'

$$g'(y) = 2x^2+3y^2-20-\frac{1}{2}x^2 = \underbrace{\frac{3x^2}{2}+3y^2-20}$$

this is a function of x and y so there is a problem...

Multiply by an integrating factor...

$$\mu(x,y)\,xy\,dx + \mu(x,y)(2x^2+3y^2-20)\,dy = 0$$

Again try to solve for f such that

$$\frac{\partial f}{\partial x} = \mu(x,y)\,xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \mu(x,y)(2x^2+3y^2-20)$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\mu(x,y)\,xy\right) = \boxed{\mu_y\,xy + \mu x}$$

$$\|$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\mu(x,y)(2x^2+3y^2-20)\right) = \boxed{\mu_x(2x^2+3y^2-20) + \mu\,4x}$$

view this as a partial differential equation in μ.

complicated...

In general

$$\mu(x,y)\,M\,dx + \mu(x,y)\,N\,dy = 0$$

$$\frac{\partial}{\partial y}\left(\mu(x,y)\,M\right) = \frac{\partial}{\partial x}\left(\mu(x,y)\,N\right)$$

all the M, M_y, N, N_y are known functions ...

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

← subscripts mean partial derivatives

try to solve for μ.

- Good: Linear in μ.
- Bad: Partial differential equation.
- Good: If we can find μ then the original problem becomes exact when multiplied by μ and so we can solve for f.

Right Now $\mu = \mu(x,y)$ ← since two variables that's where the partial derivatives came from..

Simplifying assumption that doesn't always work.

Idea

$$\boxed{\begin{array}{lll} \text{Try taking} & \mu = \mu(x) & \leftarrow \text{function of } x \text{ only} \\ \text{or} & \mu = \mu(y) & \leftarrow \text{function of } y \text{ only.} \end{array}}$$

Suppose $\mu = \mu(y)$ then what

$$\mu_y M + \mu M_y = \cancel{\mu_x} N + \mu N_x$$

$$\mu' M + \mu M_y = \mu N_x$$

linear ODE in μ.

In our example

$$\boxed{\mu' \cancel{x} y + \mu \cancel{x}} = \mu 4\cancel{x}$$

$$\mu' + \frac{\mu}{y} = \frac{4\mu}{y}$$

$$\mu' - \frac{3}{y}\mu = 0$$

← Solve for μ.

separation of variables
or variation of parameters

Separation of variables.

$$\frac{d\mu}{dy} = \frac{3}{y}\mu$$

$$\int \frac{d\mu}{\mu} = \int \frac{3}{y}\, dy$$

set zero for convenience..

$$\cancel{\ln}\mu = 3\ln y + c = \cancel{\ln} y^3$$

$$\mu(y) = y^3$$

variation of parameters

$$\mu' - \frac{3}{y}\mu = 0$$

$$e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = \frac{1}{y^3}$$

$$\underbrace{\frac{1}{y^3}\mu' - \frac{1}{y^3}\frac{3}{y}\mu}\; = 0$$

$$\left(\frac{\mu}{y^3}\right)' = 0 \qquad \frac{\mu}{y^3} = C \qquad \mu = Cy^3$$

for convenience set $C = 1$ then

same as before... → $\boxed{\mu(y) = y^3}$

$$\mu xy\,dx + \mu(2x^2+3y^2-20)\,dy = 0$$

$$y^3 xy\,dx + y^3(2x^2+3y^2-20)\,dy = 0$$

$M = xy^4$ $\qquad\qquad$ $N = y^3(2x^2+3y^2-20)$

$M_y = \boxed{4xy^3}$ $\qquad\qquad$ $N_x = \boxed{y^3(4x} + 0 - 0)$

same

$\dfrac{\partial f}{\partial x} = xy^4$ $\qquad\qquad$ $f(x,y) = \frac{1}{2}x^2y^4 + g(y)$

treat y as a constand

$$\frac{\partial f}{\partial y} = \cancel{2x^2}y^3 + g'(y) = \cancel{y^3(2x^2}+3y^2-20)$$

solve for g'

$$g'(y) = \underbrace{3y^5 - 20y^3}_{\text{only a function of } y \text{ ... much better}}$$

$$g(y) = \frac{3}{6}y^6 - \frac{20}{4}y^4 = \frac{1}{2}y^6 - 5y^4$$

Solution are given implicitly by...

$$f(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$$

Bernoulli's Equation The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{4}$$

where n is any real number, is called **Bernoulli's equation.** Note that for $n = 0$ and $n = 1$, equation (4) is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form (4) to a linear equation.

$n=2$

$u = y^{1-2} = \frac{1}{y}$

$$x\frac{dy}{dx} + y = x^2y^2.$$

$u = \frac{1}{y} \qquad y = \frac{1}{u}$

swap all the y's for u's

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{d}{du}\frac{1}{u}\right)\frac{du}{dx} = \frac{du^{-1}}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$

$$x\left(-u^{-2}\frac{du}{dx}\right) + \frac{1}{u} = x^2\left(\frac{1}{u}\right)^2$$

$$-\frac{x}{u^2}\frac{du}{dx} + \frac{1}{u} = \frac{x^2}{u^2}$$

mult by u^2

$$-x\frac{du}{dx} + u = x^2$$

linear in u . . .

solve as linear equation.

EXAMPLE 1 Solving a Homogeneous DE

comes naturally from consistency of units of measurement.

Solve $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0.$

areas. if x and y are length