Example:

$$y' = 0.1\sqrt{y} + 0.4x^2, \quad y(2) = 4.$$

$f(x,y) = 0.1\sqrt{y} + 0.4x^2$

$x_0 = 2$ $\qquad y_0 = 4$

$x_1 = 2.1$

$x_1 = x_0 + h$

Try $h = 0.1$ and then compute

$y(x_1) \approx y_1 = y_0 + hf(x_0, y_0)$

$y(2.1) \approx y_1 = 4 + 0.1\underbrace{\left(0.1\sqrt{4} + 0.4(2)^2\right)}_{1.8}$ = compute this next time... $= 4.18$

```
julia> y0=4.0
4.0

julia> x0=2.0
2.0

julia> f(x,y)=0.1*sqrt(y)+0.4*x^2
f (generic function with 1 method)

julia> f(x0,y0)
1.8

julia> h=0.1
0.1

julia> y1=y0+h*f(x0,y0)
4.18
```

$y(2.1) \approx y_1 = 4 + 0.1\left(0.1\sqrt{4} + 0.4(2)^2\right)$

$y(2.2) \approx y_2 = y_1 + hf(x_1, y_1)$

$= 4.18 + 0.1\left(0.1\sqrt{4.18} + 0.4(2.1)^2\right) \approx 4.376845...$

```
julia> x1=x0+h
2.1

julia> y2=y1+h*f(x1,y1)
4.376845048300261
```

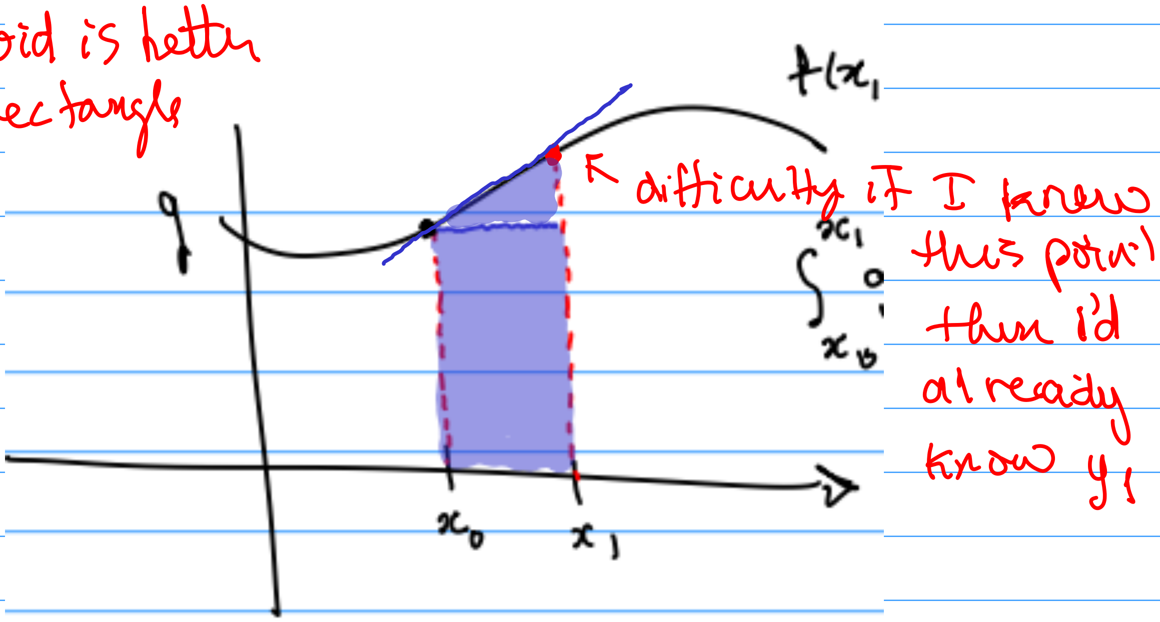
$y_{n+1} = \text{euler}(x_n, y_n) = y_n + hf(x_n, y_n)$

If $y' = f(x,y)$

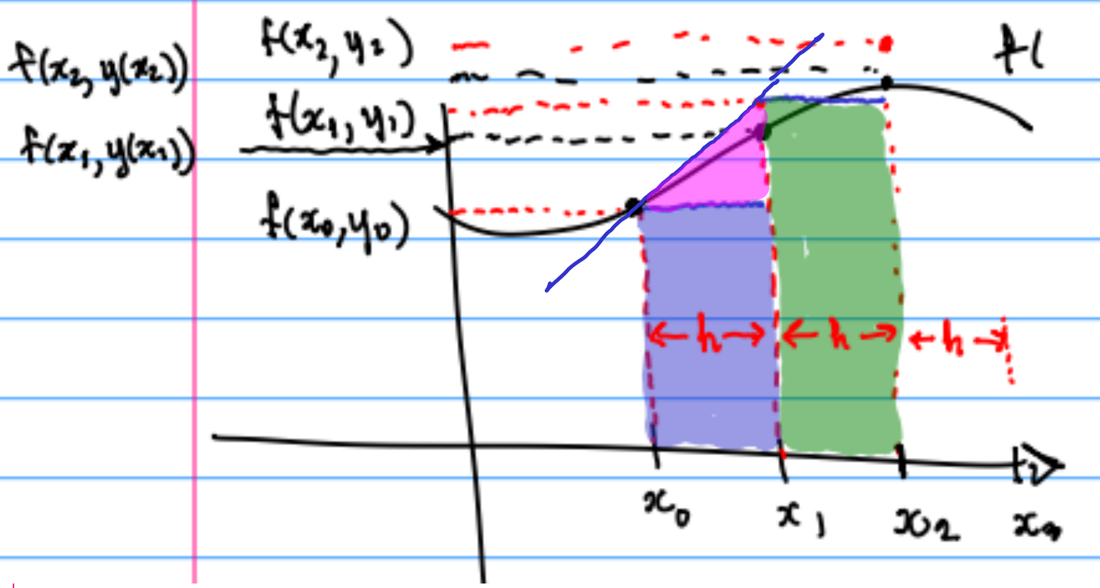
Then

$$y(x_1) - y(x_0) = \underbrace{\int_{x_0}^{x_1} f(x, y(x))\,dx}_{\text{better approximation than rectangles}}$$

Trapezoid is better than rectangle



make a better approximation using the trapezoid from the old approx.



$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x, y(x))\,dx \approx \frac{1}{2}\big(f(x_0,y_0) + f(x_1,y_1)\big)\underbrace{(x_1 - x_0)}_{h}$$

Euler approx for y_1

$$y_1 \approx y_0 + f(x_0,y_0)(x_1-x_0)$$

$$\underbrace{y_1\text{-improved} = y_0 + h\,\frac{1}{2}\Big(f(x_0,y_0) + f(x_1, \underbrace{y_0 + f(x_0,y_0)(x_1-x_0)}_{\text{Euler approx}})\Big)}_{\text{improvement}}$$

```
julia> y1imp=y0+h/2*(f(x0,y0)+f(x1,y1))
4.1884225241501305
```

compare with original approx.

```
julia> y1=y0+h*f(x0,y0)
4.18
```

Next time compare approx. to the exact solution for a diff. eq. that we know the answer for...