Math 285: Quiz 3

1. Find the general solution to the linear differential equation

$$\frac{dy}{dx} + y = e^{3x}.$$

$P = 1$

Integrating factor

$\mu = e^{\int P(x)dx} = e^{\int 1 dx} = e^{x}$

$\frac{dy}{dx}e^{x} + ye^{x} = e^{3x}e^{x}$; $\frac{d}{dx}(ye^{x}) = e^{4x}$

Therefore

$ye^{x} = \int e^{4x}dx = \frac{1}{4}e^{4x} + C$

divide by e^{x}

and

$$\boxed{y = \frac{1}{4}e^{3x} + Ce^{-x}}$$

2. Solve the exact differential equation

$df = 0$ $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$$\underbrace{(2xy^2 - 3)}_{\frac{\partial f}{\partial x}}dx + \underbrace{(2x^2y + 4)}_{\frac{\partial f}{\partial y}}dy = 0.$$

solve for f..

$\frac{\partial f}{\partial x} = 2xy^2 - 3$ $f(x,y) = x^2y^2 - 3x + g(y)$ ← y is constant

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^2 - 3x + g(y)) = 2x^2y + g'(y) = (2x^2y + 4)$

$g'(y) = 4$ $g(y) = 4y + C$ ← omit for convenience

Thus $f(x,y) = x^2y^2 - 3x + 4y$

Solutions are given implicitly by

$$\boxed{x^2y^2 - 3x + 4y = C}$$ for an arb. const. C.

General differential equation

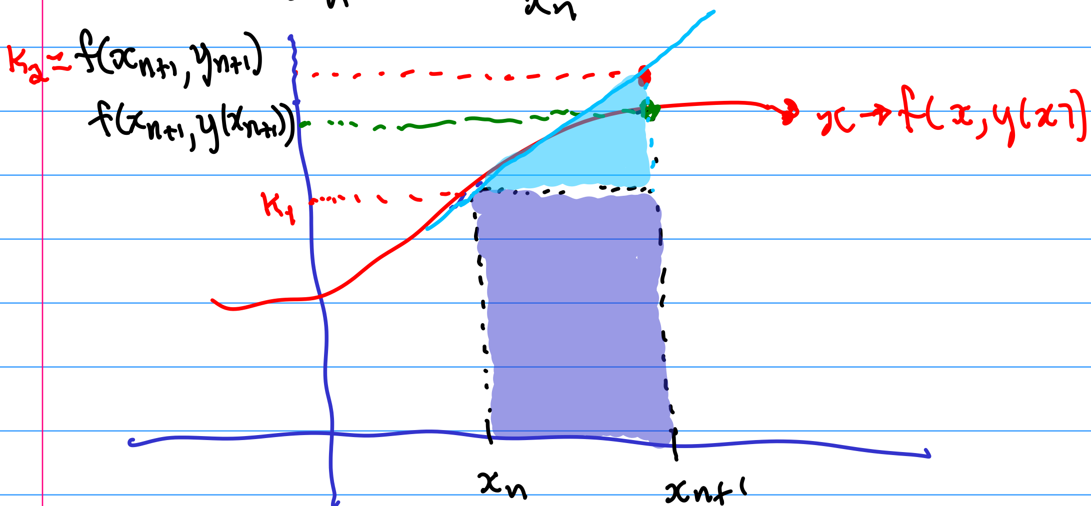
$$y' = f(x,y) \qquad y(x_0) = y_0$$

Try to approximate the solution $y(x)$.

Make a grid $x_n = x_0 + hn$

Compute y_n's so that $y_n \approx y(x_n)$.

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} y'\,dx = \int_{x_n}^{x_{n+1}} f(x,y)\,dx \approx f(x_n,y_n)\overbrace{(x_{n+1}-x_n)}^{h}$$



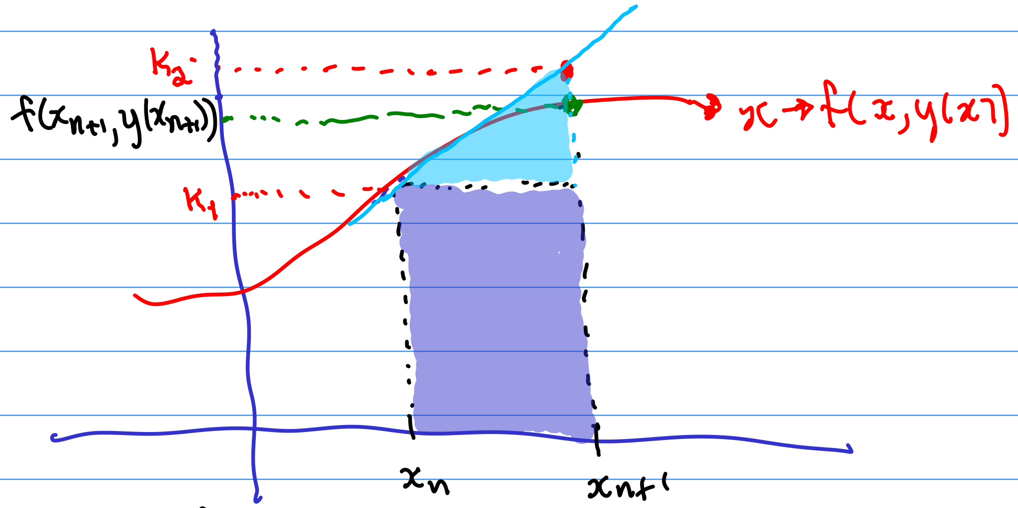
$$y_{n+1} \approx y_n + h f(x_n, y_n)$$

Then $y_{n+1} \approx y(x_{n+1})$

better approx. of integral...

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} y'\,dx = \int_{x_n}^{x_{n+1}} f(x,y)\,dx = h\left(\frac{f(x_n,y_n) + f(x_{n+1},y_{n+1})}{2}\right)$$

RK2 method or Improved Euler Method.



$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + h k_1)$$

$$y_{n+1} = y_n + h\left(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right)$$

What is the error... how good is the improvement?

Euler method

$$\boxed{y' = f(x,y)} \qquad y(x_0) = y_0$$

First step... $\quad y_1 = y_0 + h f(x_0, y_0)$

Idea plug the exact solution $y(x)$ into the method.

$$\tau = y(x_1) - y(x_0) - h f(x_0, y(x_0)) \quad \leftarrow \text{error in the method.}$$

$$= \underbrace{y(x_0+h)}_{\text{Taylor}} - y(x_0) - h f(x, y(x_0))$$

$$= \cancel{y(x_0)} + h y'(x_0) + \underbrace{\frac{h^2}{2} y''(c)}_{\text{remainder.}} - \cancel{y(x_0)} - h f(x, y(x_0))$$

$$= \frac{h^2}{2} y''(c) \quad \text{error...}$$

h^2 times some number...

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + h k_1)$$

$$y_{n+1} = y_n + h\left(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right)$$

To make things easier — and because it's actually enough to draw general conclusions —

$$y' = f(y) \qquad\qquad y(x_0) = y_0.$$

↑ no x here

$$k_1 = f(y_n)$$

$$k_2 = f(y_n + h k_1)$$

$$y_{n+1} = y_n + h\left(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right)$$

$$\tau = y(x_0+h) - y(x_0) - h\left(\tfrac{1}{2}f(y_0) + \tfrac{1}{2}f\big(y_0 + h f(y_0)\big)\right)$$

$$= \cancel{y(x_0)} + h y'(x_0) + \frac{h^2}{2}y''(x_0) + \underbrace{\frac{h^3}{3!}y'''(c_1)}_{\text{remainder}} - \cancel{y(x_0)}$$

$$- h\left(\tfrac{1}{2}f(y_0) + \tfrac{1}{2}f\big(y_0 + h f(y_0)\big)\right)$$

$$f\big(y_0 + h f(y_0)\big) = f(y_0) + h f(y_0) f'(y_0) + \underbrace{\frac{(h f(y_0))^2}{2} f''(c_2)}_{\text{remainder}}.$$

$f(y(x_0))$

$$\tau = h \cancel{y'(x_0)} + \frac{h^2}{2}y''(x_0) + \frac{h^3}{3!}y'''(c_1)$$

$$- h\left(\cancel{\tfrac{1}{2}f(y_0)} + \tfrac{1}{2}\left(\cancel{f(y_0)} + h f(y_0) f'(y_0) + \frac{(h f(y_0))^2}{2} f''(c_2)\right)\right)$$

$$= \boxed{\frac{h^2}{2}} \underbrace{y''(x_0)}_{f'(y_0)f(y_0)} + \frac{h^3}{3!}y'''(c_1) - \boxed{\frac{h}{2}\left[h f(y_0) f'(y_0)\right.} + \frac{(h f(y_0))^2}{2} f''(c_2)\Big)$$

$$y' = f(y)$$

$$y'' = \frac{d}{dx}\big(f(y)\big) = f'(y)\frac{dy}{dx} = f'(y)\, f(y)$$

$$\tau = \frac{h^3}{3!} y'''(c_1) - \frac{h}{2}\left(\frac{(h f(y_0))^2}{2} f''(c_2)\right)$$

$$= h^3 \Big(\underbrace{\frac{1}{3!} y'''(c_1) - \frac{1}{4} f(y_0)^2 f''(c_2)}_{\text{some number.}}\Big)$$

$\boxed{h^3 \text{ gets small very fast compared to } h^2 \text{ as } h \to 0.}$