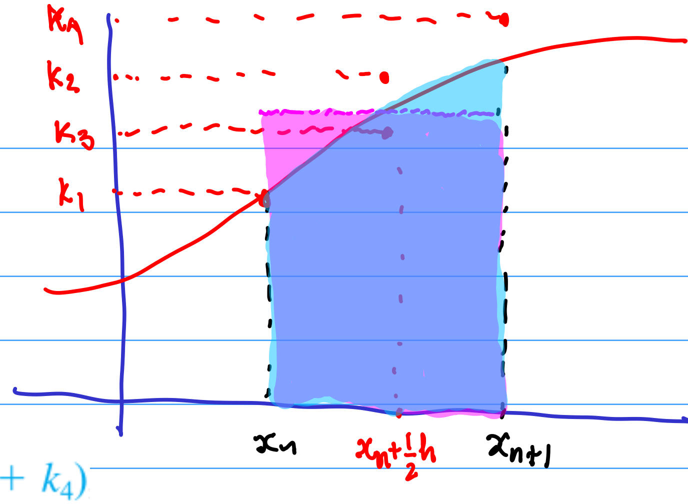
Cartoon



$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right)$$
$$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2\right)$$
$$k_4 = f(x_n + h, y_n + hk_3).$$

$$y_{n+1} = y_n + \underbrace{\frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

Some average of the heights of k_1, k_2, k_3, k_4 that is use to approximate the area.

Weighted average is supposed to be a good approximation of the area under the curve.

$\tau = h^5$ (some constant)

as h gets small then h^5 gets very small...

In Problems 3–12 use the RK4 method with $h = 0.1$ to obtain a four-decimal approximation of the indicated value.

$x_n = x_0 + nh$

5. $y' = 1 + y^2$, $y(0) = 0$; $y(0.5)$

$f(\cancel{x}, y) = 1 + y^2$ $\quad y_0 = 0$

$x_0 = 0$
$x_1 = 0.1$
$x_2 = 0.2$
$x_3 = 0.3$
$x_4 = 0.4$
$x_5 = 0.5$

$$k_1 = f(x_n, y_n) =$$
$$k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right)$$
$$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2\right)$$
$$k_4 = f(x_n + h, y_n + hk_3).$$

On a quiz at most one step and probably Euler and then either RK4 or $\underbrace{\text{Improved Euler}}_{\text{RK2}}$

```
julia> f(x,y)=1+y^2
f (generic function wi

julia> h=0.1
0.1

julia> x0=0.0
0.0

julia> y0=0.0
0.0
```

```
julia> k1=f(x0,y0)
1.0

julia> k2=f(x0+1/2*h,y0+1/2*h*k1)
1.0025

julia> k3=f(x0+1/2*h,y0+1/2*h*k2)
1.002512515625

julia> k4=f(x0+h,y0+h*k3)
1.0100503134398477
```

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

```
julia> y1=y0+h/6*(k1+2*k2+2*k3+k4)
0.10033458907816414
```

5. $y' = 1 + y^2, y(0) = 0; \quad y(0.5)$

$y(x_1) \approx y_1$

$y(0.1) \approx$ 0.10033458907816414

Next step.

```
julia> k1=f(x1,y1)
1.0100670297654841

julia> k2=f(x1+1/2*h,y1+1/2*h*k1)
1.0227520843143243

julia> k3=f(x1+1/2*h,y1+1/2*h*k2)
1.022943825341258 6

julia> k4=f(x1+h,y1+h*k3)
1.0410585001366541
```

```
julia> y2=y1+h/6*(k1+2*k2+2*k3+k4)
0.2027098782317192
```

$y(x_2) \approx y_2$

$y(0.2) \approx$ 0.2027098782317192

some calculation like this on quiz 5.

Note RK4 is a good tradeoff between a complex algorithm and an accurate one... so we don't go any further...

4.1 PRELIMINARY THEORY—LINEAR EQUATIONS

Simple example: $9y'' - 12y' + 4y = 0$

second order linear differential equation with constant coefficients

already done: first order linear differential equation with any coefficients

$$9y'' - 12y' + 4y = 0$$

introduce new variable $z = y'$ then $z' = y''$

$$9z' - 12z + 4y = 0$$
$$-y' + z = 0$$

$$z' - \frac{12}{9}z + \frac{4}{9}y = 0$$
$$y' - z = 0$$

Vectors

$$\underbrace{\begin{bmatrix} y' \\ z' \end{bmatrix}}_{Y} = \begin{bmatrix} 0y + 1z \\ -4/9\, y + 12/9\, z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -4/9 & 12/9 \end{bmatrix}}_{A} \begin{bmatrix} y \\ z \end{bmatrix}$$

Vector differential equation

$$Y' = AY$$
$$Y = e^{Ax} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

scalar case:

$$y' = 3y$$
$$y = ce^{3x}$$

Take away messages:

- Can convert higher order diff. eqns. to first order by introducing more variables
- The answer to the original problem had something to do with exponential function

$$9y'' - 12y' + 4y = 0$$

Guess and check method.

Guess: $y = e^{rx}$ ← whole class of exponentials...

Check. $y' = re^{rx}$

$y'' = r^2 e^{rx}$

$$9r^2\cancel{e^{rx}} - 12r\cancel{e^{rx}} + 4\cancel{e^{rx}} = 0$$

$$9r^2 - 12r + 4 = 0$$

$$(3r-2)(3r-2) = 0$$

$$r = 2/3 \quad (\text{mult. } 2.)$$

Solution: $y = e^{\frac{2x}{3}}$