Higher order linear differential equations.

Solve: $$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

Subject to: $y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$

all these are functions of x.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g$$

assume $a_0, a_1, \ldots, a_n$ are all continuous functions of x
and also g is a continuous function of x.

If I add conditions:

$y(x_0) = y_0,\ y'(x_0) = y_1, \ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$ then..

THEOREM 4.1.1 Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

We already know the first order equation $y' = f(x, y)$
such that $y(x_0) = y_0$
gave solutions...

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g$$

$v_0 = y,\ v_1 = y',\ v_2 = y'' \ldots\ v_{n-1} = y^{(n-1)}$ $\quad v_{n-1}' = y^{(n)}$

$$a_n v_{n-1}' + a_{n-1} v_{n-1} + \cdots + a_1 v_1 + a_0 v_0 = g$$

$$v_0 = y, \quad v_1 = y', \quad v_2 = y'' \quad \ldots \quad v_{n-1} = y^{(n-1)}$$

$$v_1 = v_0', \quad v_2 = v_1' \quad \ldots \quad v_{n-1} = v_{n-2}'$$

System of n equations

$$\begin{cases} v_0' = v_1 \\ v_1' = v_2 \\ \vdots \\ v_{n-2}' = v_{n-1} \\ v_{n-1}' = \dfrac{g}{a_n} - \dfrac{a_{n-1}}{a_n} v_{n-1} - \cdots - \dfrac{a_1}{a_n} v_1 - \dfrac{a_0}{a_n} v_0 \end{cases}$$

$$V = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix}, \quad F(x,V) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ \dfrac{g}{a_n} - \dfrac{a_{n-1}}{a_n} v_{n-1} - \cdots - \dfrac{a_1}{a_n} v_1 - \dfrac{a_0}{a_n} v_0 \end{bmatrix}$$

Then the original system is

$$V' = F(x, V).$$

vector version of the same thing, so need

Know this exists and unique

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

conditions

$$V(x_0) = V_0$$

to make unique..

$$\begin{bmatrix} v_0(x_0) \\ v_1(x_0) \\ \vdots \\ v_{n-1}(x_0) \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

conditions to make unique

That exactly the same condition as ...

$y(x_0)=y_0,\ y'(x_0)=y_1,\ \dots,\ y^{(n-1)}(x_0)=y_{n-1}$

DEFINITION 4.1.1 Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent.**

DEFINITION 4.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \dots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

THEOREM 4.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \dots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval.

$$9y'' - 12y' + 4y = 0$$

want a unique solution then I need two conditions

$y(x_0) = y_0$ and $y'(x_0) = y_1$

$y(0) = 1$ $\qquad$ $y'(0) = 2$ ← example

Know this family of solutions

$$y = c_1 e^{\frac{2}{3}x}$$

Try solving for constant

$$y(0) = c_1 e^{\frac{2}{3}\cdot 0} = c_1 = 1$$

don't match... problem... need more constants to solve for..

Need to find more solutions..

Also $y' = c_1 \frac{2}{3} e^{\frac{2}{3}x}$

$$y'(0) = \frac{2}{3} e^{\frac{2}{3}\cdot 0} = \frac{2}{3}$$

Let $w = u e^{\frac{2}{3}x}$ and substitute w into the diff. eq.

$$9y'' - 12y' + 4y = 0$$

$$9w'' - 12w' + 4w = 0$$

$$w = u e^{\frac{2}{3}x}$$

$$w' = u' e^{\frac{2}{3}x} + u \frac{2}{3} e^{\frac{2}{3}x}$$

$$w'' = u'' e^{\frac{2}{3}x} + u' \frac{2}{3} e^{\frac{2}{3}x} + u' \frac{2}{3} e^{\frac{2}{3}x} + u \frac{4}{9} e^{\frac{2}{3}x}$$

$$w'' = u'' e^{\frac{2}{3}x} + \frac{4}{3} u' e^{\frac{2}{3}x} + \frac{4}{9} u e^{\frac{2}{3}x}$$

$$9w'' = 9u''e^{\frac{2}{3}x} + 12u'e^{\frac{2}{3}x} + 4ue^{\frac{2}{3}x}$$
$$-12w' = -12u'e^{\frac{2}{3}x} - 8ue^{\frac{2}{3}x}$$
$$4w = 4ue^{\frac{2}{3}x}$$

$$9w'' - 12w' + 4w = 9u''e^{\frac{2}{3}x} + \cancel{12u'e^{\frac{2}{3}x}} \cancel{-12u'e^{\frac{2}{3}x}}$$

$$0 = \cancel{9}u''\cancel{e^{\frac{2}{3}x}}$$

since w is supposed to satisfy the diff. eq.

$$9w'' - 12w' + 4w = 0$$

$$u'' = 0$$

so $u(x) = ax + b$

$$w(x) = u(x)e^{\frac{2}{3}x}$$

$$w(x) = (ax+b)e^{\frac{2}{3}x}$$

two constants

$c_2 = a$
$c_1 = b$

$$y(x) = c_1 e^{\frac{2}{3}x} + \boxed{c_2 x e^{\frac{2}{3}x}}$$

family of solution

another solution.

Original problem:

$$9y'' - 12y' + 4y = 0$$

$\boxed{y(0) = 1}$ $\qquad y'(0) = 2$

$$y(0) = c_1 e^{\frac{2}{3}\cdot 0} + c_2 \cdot 0 \cdot e^{\frac{2}{3}\cdot 0} = c_1 = 1$$

$$y'(x) = \frac{d}{dx}\left(c_1 e^{\frac{2}{3}x} + c_2 x e^{\frac{2}{3}x}\right) = \frac{2}{3}c_1 e^{\frac{2}{3}x} + c_2 e^{\frac{2}{3}x} + \frac{2}{3}c_2 x e^{\frac{2}{3}x}$$

$$y'(0) = \frac{2}{3}c_1 e^{\frac{2}{3}\cdot 0} + c_2 e^{\frac{2}{3}\cdot 0} + \frac{2}{3}c_2 0 e^{\frac{2}{3}\cdot 0} = \frac{2}{3}\cdot 1 + c_2 = 2$$

$$c_2 = 2 - \frac{2}{3} = \frac{4}{3}$$

Unique solution:

$$\boxed{y(x) = 1\, e^{\frac{2}{3}x} + \frac{4}{3}\, x e^{\frac{2}{3}x}}$$

Reflect there were two solutions

$$y_1(x) = e^{\frac{2}{3}x} \quad \text{and} \quad y_2(x) = x e^{\frac{2}{3}x}$$

and the general solution was

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

are these constants enough to satisfy any conditions of the form

$$\boxed{y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1}$$

YES!

$$W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} e^{\frac{2}{3}x} & x e^{\frac{2}{3}x} \\ \frac{2}{3} e^{\frac{2}{3}x} & e^{\frac{2}{3}x} + \frac{2}{3} x e^{\frac{2}{3}x} \end{bmatrix}$$

$$= e^{\frac{2}{3}x}\left(e^{\frac{2}{3}x} + \frac{2}{3} x e^{\frac{2}{3}x}\right) - x e^{\frac{2}{3}x}\left(\frac{2}{3} e^{\frac{2}{3}x}\right)$$

$$= e^{\frac{2}{3}x}\, e^{\frac{2}{3}x} = e^{\frac{4}{3}x} \neq 0 \text{ for any } x$$

but actually it is enough to check for only one x that it's not zero, because that guarantees not zero for all the other x's.