$$y'' + 4y' + 7y = 0$$

$y = e^{rx}$ plug it in... $y' = re^{rx}$ $y'' = r^2 e^{rx}$

$$r^2 \cancel{e^{rx}} + 4r\cancel{e^{rx}} + 7\cancel{e^{rx}} = 0\,\cancel{e^{rx}}$$

$$r^2 + 4r + 7 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 28}}{2} = -2 \pm \frac{\sqrt{-12}}{2}$$

$$= -2 \pm \sqrt{-3} = -2 \pm i\sqrt{3}$$

solutions to the differential equation are

$$y_1 = e^{(-2 + i\sqrt{3})x} \qquad y_2 = e^{(-2 - i\sqrt{3})x}$$

one solution for each root of the quadratic equation...

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \ldots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$$

$2! = 1 \cdot 2$

$3! = 1 \cdot 2 \cdot 3$

$4! = 1 \cdot 2 \cdot 3 \cdot 4$

plug in $x = i\theta$

$$e^{i\theta} = 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \frac{1}{6!}(i\theta)^6 + \frac{1}{7!}(i\theta)^7 + \ldots$$

$$= 1 + i\theta + \frac{1}{2!}i^2\theta^2 + \frac{1}{3!}i^3\theta^3 + \frac{1}{4!}i^4\theta^4 + \frac{1}{5!}i^5\theta^5 + \frac{1}{6!}i^6\theta^6 + \frac{1}{7!}i^7\theta^7 + \ldots$$

$$= 1 + i\theta + \frac{1}{2!}(-1)\theta^2 + \frac{1}{3!}(-i)\theta^3 + \frac{1}{4!}\cdot 1\,\theta^4 + \frac{1}{5!}i\,\theta^5 + \frac{1}{6!}(-1)\theta^6 + \frac{1}{7!}(-i)\theta^7 + \ldots$$

$$= 1 + \frac{1}{2!}(-1)\theta^2 + \frac{1}{4!}\cdot 1\,\theta^4 + \frac{1}{6!}(-1)\theta^6 + \dots$$
$$+ i\theta + \frac{1}{3!}(-i)\theta^3 + \frac{1}{5!}i\,\theta^5 + \frac{1}{7!}(-i)\theta^7 + \dots$$

$$= 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \dots$$
$$+ i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \dots\right)$$

Therefore,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

recall

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

What do those solutions mean?

$$y_1 = e^{(-2+i\sqrt{3})x} \qquad y_2 = e^{(-2-i\sqrt{3})x}$$

$$y'' + 4y' + 7y = 0$$

If y_1 and y_2 are solutions, then...

$$y_1'' + 4y_1' + 7y_1 = 0 \qquad y_2'' + 4y_2' + 7y_2 = 0$$

$$c_1 y_1'' + c_1 4 y_1' + c_1 7 y_1 = c_1 0 \qquad c_2 y_2'' + c_2 4 y_2' + c_2 7 y_2 = c_2 0$$
$$(c_1 y_1)'' + 4(c_1 y_1)' + 7(c_1 y_1) = 0 \qquad (c_2 y_2)'' + 4(c_2 y_2)' + 7(c_2 y_2) = 0$$

Now add the solutions:

$$(c_1 y_1)'' + 4(c_1 y_1)' + 7(c_1 y_1) = 0$$
$$(c_2 y_2)'' + 4(c_2 y_2)' + 7(c_2 y_2) = 0$$

$$(c_1 y_1 + c_2 y_2)'' + 4(c_1 y_1 + c_2 y_2)' + 7(c_1 y_1 + c_2 y_2) = 0$$

Thus $y = c_1 y_1 + c_2 y_2$ is a solution for all values of c_1 and c_2.

$$y = c_1 e^{(-2+i\sqrt{3})x} + c_2 e^{(-2-i\sqrt{3})x}$$

recall

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{(-2+i\sqrt{3})x} = e^{-2x} e^{i\sqrt{3}x} = e^{-2x}\left(\cos\sqrt{3}x + i\sin\sqrt{3}x\right)$$

cosine is even

sine is odd.

$$e^{(-2-i\sqrt{3})x} = e^{-2x} e^{-i\sqrt{3}x} = e^{-2x}\left(\cos(-\sqrt{3}x) + i\sin(-\sqrt{3}x)\right)$$

$$e^{(-2-i\sqrt{3})x} = e^{-2x}\left(\cos\sqrt{3}x - i\sin\sqrt{3}x\right)$$

$$y = c_1 e^{-2x}\left(\cos\sqrt{3}x + i\sin\sqrt{3}x\right) + c_2 e^{-2x}\left(\cos\sqrt{3}x - i\sin\sqrt{3}x\right)$$

$$y'' + 4y' + 7y = 0$$

Conditions for a unique solution to be $y(x_0) = y_0$ and $y'(x_0) = y_1$

Note to solve for c_1 and c_2 we plug the solution into the conditions. What happens? If y_0 and y_1 are real numbers then solving for c_1 and c_2 give complex numbers because of the i in the definition of y.

Idea factor out the cos and sin seperately

$$y = e^{-2x}\left(c_1\cos\sqrt{3}x + c_2\cos\sqrt{3}x\right) + e^{-2x}\left(c_1 i\sin\sqrt{3}x - c_2 i\sin\sqrt{3}x\right)$$

$$= \underbrace{e^{-2x}\cos\sqrt{3}x}_{\text{real function}}\underbrace{(c_1+c_2)}_{\text{constant } A} + \underbrace{e^{-2x}\sin\sqrt{3}x}_{\text{real funct.}}\underbrace{(ic_1 - ic_2)}_{\text{different constant } B}$$

$$= Ae^{-2x}\cos\sqrt{3}x + Be^{-2x}\sin\sqrt{3}x$$

Now if y_0 and y_1 are real numbers then solving for A and B in

$$y = Ae^{-2x}\cos\sqrt{3}\,x + Be^{-2x}\sin\sqrt{3}\,x$$

such that $y(x_0)=y_0$ and $y'(x_0)=y_1$ gives constants A and B which are also real.

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4. $y'' - 3y' + 2y = 0$ ← zero on right side

↑ constant coefficients

linear homogeneous 2nd order differential equation with constant coefficients...

Guess $y = e^{rx}$ plug in $y' = re^{rx}$ $\quad y'' = r^2e^{rx}$

$$r^2\cancel{e^{rx}} - 3r\cancel{e^{rx}} + 2\cancel{e^{rx}} = 0$$

$$r^2 - 3r + 2 = 0$$

$a = 1 \quad b = -3 \quad c = 2$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \{1, 2\}$$

$y_1 = e^{1x} \qquad y_2 = e^{2x}$

General solution

$$y = c_1e^{x} + c_2e^{2x}$$

problem didn't have a condition to find a unique solution..