6. $y'' - 8y' + 20y = \overbrace{100x^2 - 26xe^x}^{g}$

g_1 g_2

First work the homogeneous problem:

$$y_h'' - 8y_h' + 20y_h = 0$$

Guess $y_h = e^{rx}$ plug in

$$r^2e^{rx} - 8re^{rx} + 20e^{rx} = 0e^{rx}$$

$$r^2 - 8r + 20 = 0$$

$a=1 \quad b=-8 \quad c=20$

80
64
16

Quadratic formula

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{8 \pm \sqrt{64-80}}{2} = \frac{8 \pm i\sqrt{16}}{2} = 4 \pm 2i$$

Homogeneous solution

$$y_h = c_1 e^{(4+2i)x} + c_2 e^{(4-2i)x}$$

alternatively

$$y_h = c_1 e^{4x}\cos 2x + c_2 e^{4x}\sin 2x$$

Now solve the first inhomogeneous problem. How?

$$y_{p_1}'' - 8y_{p_1}' + 20y_{p_1} = 100x^2$$

Guess $y_h = Ax^2 + Bx + C$ plug in

$$y = Ax^2 + Bx + C$$
$$y' = 2Ax + B$$
$$y'' = 2A$$

$$y'' = 2A$$
$$-8y' = -16Ax - 8B$$
$$20y = 20Ax^2 + 20Bx + 20C$$

$$100x^2 = 20Ax^2 + (20B - 16A)x + 20C - 8B + 2A$$

equating coefficients of powers of x

System of linear equations

$$\begin{cases} 100 = 20A \\ 0 = 20B - 16A \\ 0 = 20C - 8B + 2A \end{cases}$$

$A = 5$

$B = \frac{16 \cdot A}{20} = \frac{4}{5} \cdot 5 = 4$

$C = \frac{8B - 2A}{20} = \frac{2}{5}B - \frac{1}{10}A$

$= \frac{8}{5} - \frac{5}{10} = \frac{16-5}{10} = \frac{11}{10}$

Solution

$$y_{p_1} = 5x^2 + 4x + \frac{11}{10}$$

Solve other inhomogeneous equation

$$y_{p_2}'' - 8y_{p_2}' + 20y_{p_2} = -26xe^x$$

Guess $y = (Ax + B)e^x$ plug it in

Solve for A, B

$$y_{p_2} = \left(-\frac{20}{13}x - \frac{120}{169}\right)e^x$$

General solution

$$y = c_1 e^{4x}\cos 2x + c_2 e^{4x}\sin 2x + 5x^2 + 4x + \frac{11}{10} + \left(-\frac{20}{13}x - \frac{120}{169}\right)e^x$$