4.6 Variation of Parameters

Constant coefficients: Guess the answer as $y = e^{rx}$

Simplest case of variation of parameters

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$y(x) = u(x)e^{-\int P(x)dx}$$

$$y(x) = u(x)y_h(x)$$

solution to the homogeneous problem

integrating factor idea ..

$$\mu = e^{\int P(x)dx}$$

and multiply both sides by μ.

Next more complicated case:

$$\boxed{y'' + P(x)y' + Q(x)y = f(x)}$$

plug in here

Suppose we can solve the homogeneous problem

$$y'' + P(x)y' + Q(x)y = 0$$

general solution to homogeneous problem.

then there is y_1 and y_2 and $y_h = C_1y_1 + C_2y_2$

Plug this in

use homogeneous solution to simplify ..

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y'(x) = u_1'(x)y_1(x) + u_1(x)y_1'(x) + u_2'(x)y_2(x) + u_2(x)y_2'(x)$$

$$= u_1'(x)y_1(x) + u_2'(x)y_2(x) + u_1(x)y_1'(x) + u_2(x)y_2'(x)$$

$$y''(x) = \frac{d}{dx}\left[u_1'(x)y_1(x) + u_2'(x)y_2(x)\right] +$$

$$u_1'(x)y_1'(x) + u_1(x)y_1''(x) + u_2'(x)y_2'(x) + u_2(x)y_2''(x)$$

$$= \frac{d}{dx}\left[u_1'(x)y_1(x) + u_2'(x)y_2(x)\right] +$$

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) + u_1(x)y_1''(x) + u_2(x)y_2''(x)$$

$$\frac{d}{dx}\left[u_1'(x)y_1(x) + u_2'(x)y_2(x)\right] +$$

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) + u_1(x)\boxed{y_1''(x)} + u_2(x)\boxed{y_2''(x)}$$

$$+ P(x)\left(u_1'(x)y_1(x) + u_2'(x)y_2(x) + u_1(x)\boxed{y_1'(x)} + u_2(x)\boxed{y_2'(x)}\right)$$

$$+ Q(x)\left(u_1(x)\boxed{y_1(x)} + u_2(x)\boxed{y_2(x)}\right) = f(x)$$

The terms in $\boxed{}$ and $\boxed{}$ cancel since y_1 and y_2 solve the homogeneous equation.

$$\frac{d}{dx}\left[u_1'(x)y_1(x) + u_2'(x)y_2(x)\right] + u_1'(x)y_1'(x) + u_2'(x)y_2'(x)$$

$$+ P(x)\left(u_1'(x)y_1(x) + u_2'(x)y_2(x)\right) = f(x)$$

or

$$\frac{d}{dx}\left[u_1'(x)y_1(x) + u_2'(x)y_2(x)\right] + P(x)\left(u_1'(x)y_1(x) + u_2'(x)y_2(x)\right)$$

$$+ u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x)$$

Since there are two functions u_1 and u_2 to solve for, I can afford an extra condition to simplify things..

$$\left.\begin{aligned} u_1'(x)y_1(x) + u_2'(x)y_2(x) &= 0 \\ u_1'(x)y_1'(x) + u_2'(x)y_2'(x) &= f(x) \end{aligned}\right\} \text{now I have two equations in two unknowns...}$$

Write in matrix form

$$\underbrace{\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}}_{\text{Same matrix used in the definition of } W.} \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$W = \det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

Use Cramer's rule to solve 2×2 matrix equation.

$$W_1 = \det\begin{bmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{bmatrix} \qquad W_2 = \det\begin{bmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{bmatrix}$$

answer $\begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} W_1/W \\ W_2/W \end{bmatrix} = \begin{bmatrix} -y_2(x)f(x)/W \\ y_1(x)f(x)/W \end{bmatrix}$

$$W_1 = \det\begin{bmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{bmatrix} = 0 - y_2(x)f(x)$$

$$W_2 = \det\begin{bmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{bmatrix} = y_1(x)f(x) - 0$$

Thus

$$u_1'(x) = \frac{-y_2(x)f(x)}{W} \quad \text{and} \quad u_2'(x) = \frac{y_1(x)f(x)}{W}$$

So once you solve the homogeneous problem to find y_1 and y_2 then u_1 and u_2 can be obtained as

$$u_1(x) = \int -\frac{y_2(x)f(x)}{W}\,dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)f(x)}{W}\,dx$$

and the solution to the inhomogeneous problem is $y = u_1y_1 + u_2y_2$

23. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{3/2};$

$y_1 = x^{-1/2}\cos x,\ y_2 = x^{-1/2}\sin x$

$$u_1(x) = \int -\frac{y_2(x)\, f(x)}{W}\, dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)\, f(x)}{W}\, dx$$

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} \dfrac{\cos x}{\sqrt{x}} & \dfrac{\sin x}{\sqrt{x}} \\ \dfrac{-(\sin x)\sqrt{x} - (\cos x)\frac{1}{2\sqrt{x}}}{x} & \dfrac{(\cos x)\sqrt{x} - (\sin x)\frac{1}{2\sqrt{x}}}{x} \end{bmatrix}$$

$$= \frac{\cos^2 x}{x} + \frac{\sin^2 x}{x} = \frac{1}{x}$$

$$u_1(x) = \int -\frac{\frac{\sin x}{\sqrt{x}} \cdot x^{3/2}}{1/x}\, dx = -\int x^2 \sin x\, dx$$

$$u_2(x) = \int \frac{\frac{\cos x}{\sqrt{x}} \cdot x^{3/2}}{1/x}\, dx = \int x^2 \cos x\, dx$$

Finish next time