# Cauchy-Euler Equations

$$x^2y'' - 2xy' - 4y = 0$$

**Cauchy-Euler Equation** A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

n derivatives (sort of) looses n powers of $x$ by the power rule

put the powers of $x$ back by multiplying by $x^n$

Dimensional arguments indicate how to solve and also explain why the problems happen in applications...

$$a_n x^n \frac{d^n y}{dx^n} +$$

dimensionally $x^n$ is downstairs

$x^n$ is upstairs

if $x$ had dimensional units of measurement they cancel...

Example

$$x^2y'' - 2xy' - 4y = 0$$

Since has something to do with power rule try $y = x^n$

$$y = x^n, \quad y' = nx^{n-1} \quad y'' = n(n-1)x^{n-2}$$

$$x^2\, n(n-1)x^{n-2} - 2x\, nx^{n-1} - 4x^n = 0$$

$$(n(n-1) - 2n - 4)\,x^n = 0$$

if $x \neq 0$ then

$$n(n-1) - 2n - 4 = 0$$

$$n^2 - n - 2n - 4 = 0$$

$$n^2 - 3n - 4 = 0$$

$$(n-4)(n+1) = 0 \quad \text{so} \quad n = 4, -1$$

I found 2 ind solutions

$$y_1 = x^4 \quad , \quad y_2 = x^{-1}$$

general solution to $x^2 y'' - 2xy' - 4y = 0$ is

$$y = c_1 x^4 + c_2 x^{-1}$$

#23. $x^2 y'' + xy' - y = \ln x$ $\quad f(x) = \ln x$

Solve homogeneous problem and then use variation of parameters to solve the inhomogeneous problem.

$$y = x^n, \quad y' = n x^{n-1} \quad y'' = n(n-1) x^{n-2}$$

$$x^2 n(n-1) x^{n-2} + x n x^{n-1} - x^n = 0$$

$$n(n-1) + n - 1 = 0$$

$$n^2 - \cancel{n} + \cancel{n} - 1 = 1$$

$$n^2 = 1 \quad \text{so} \quad n = 1, -1$$

General solution to homogeneous problem $y_h = c_1 x + c_2 x^{-1}$.

Use variation of parameters...

$y_1 = x \quad y_2 = x^{-1}$

$f(x) = \ln x$

$$u_1'(x) = \frac{-y_2(x) f(x)}{W} \quad \text{and} \quad u_2'(x) = \frac{y_1(x) f(x)}{W}$$

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{bmatrix} = -x^{-1} - x^{-1} = -2x^{-1}$$

$$u_1'(x) = \frac{-x^{-1}\ln x}{-2x^{-1}} = \frac{1}{2}\ln x$$

$$u_2'(x) = \frac{x\ln x}{-2x^{-1}} = -\frac{x^2}{2}\ln x$$

$$u_1 = \int \frac{1}{2}\ln x\,dx = \frac{1}{2}\int \ln x\,dx = \frac{1}{2}\left(uv - \int v\,du\right)$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$= \frac{1}{2}\left(x\ln x - \int x\frac{1}{x}dx\right) = \frac{1}{2}(x\ln x - x)$$

$$u_2 = \int -\frac{x^2}{2}\ln x\,dx = -\frac{1}{2}\int x^2\ln x\,dx$$

$u = \ln x \qquad dv = x^2dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{3}x^3$

$$= -\frac{1}{2}\left(\frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3\cdot\frac{1}{x}dx\right) = -\frac{1}{6}x^3\ln x + \frac{1}{18}x^3$$

$$y_p = u_1y_1 + u_2y_2 = \frac{1}{2}(x\ln x - x)x + \left(-\frac{1}{6}x^3\ln x + \frac{1}{18}x^3\right)x^{-1}$$

General solution to the original (inhomogeneous) problem

$$y = c_1x + c_2x^{-1} + \frac{1}{2}(x\ln x - x)x + \left(-\frac{1}{6}x^3\ln x + \frac{1}{18}x^3\right)x^{-1}$$

# 7 The Laplace Transform

**DEFINITION 7.1.1 Laplace Transform**

Let $f$ be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\}(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad (2)$$

is said to be the **Laplace transform** of $f$, provided that the integral converges.

maps a funtion of $t$ into a function of $s$.

Examples $f(t)=1$.

$$\mathscr{L}\{1\}(s) = \int_0^\infty e^{-st} \cdot 1\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty$$

$$= \lim_{t\to\infty} \left(-\frac{1}{s} e^{-st}\right) + \frac{1}{s} e^{-s\cdot 0}$$

If $s>0$ then

$$\mathscr{L}\{1\}(s) = \frac{1}{s}$$ if $s=0$ trouble...

If $s<0$ the integral diverges so $\mathscr{L}\{1\}(s)$ doesn't make sense for $s<0$.

If $s=0$ then $\mathscr{L}\{1\}(0) = \int_0^\infty e^{-0\cdot t} \cdot 1\, dt = \int_0^\infty 1\, dt = \infty$

So again $\mathscr{L}\{1\}(s)$ diverges if $s \leq 0$.

Example $f(t)=t$

$$\mathcal{L}\{t\}(s)=\int_0^\infty e^{-st}\cdot t\,dt = \left(uv-\int v\,du\right)$$

$$u=t \qquad dv=e^{-st}\,dt$$

$$du=dt \qquad v=-\frac{1}{s}e^{-st}$$

$$=-\frac{t}{s}e^{-st}\Big|_0^\infty+\int_0^\infty \frac{1}{s}e^{-st}\,dt=\left(-\frac{t}{s}e^{-st}-\frac{1}{s^2}e^{-st}\right)\Big|_0^\infty$$

$$=\lim_{t\to\infty}\left(-\frac{t}{s}e^{-st}-\frac{1}{s^2}e^{-st}\right)-\left(\cancel{-\frac{0}{s}e^{-s0}}-\frac{1}{s^2}e^{-s\cdot 0}\right)$$

If $s>0$ then

$$\mathcal{L}\{t\}(s)=\frac{1}{s^2}$$

What is $\mathcal{L}\{t^n\}(s)$ for $s>0$ ?

$$\mathcal{L}\{t^n\}(s)=\int_0^\infty e^{-st}\cdot t^n\,dt = \left(uv-\int v\,du\right)$$

$$u=t^n \qquad dv=e^{-st}\,dt$$

$$du=nt^{n-1}\,dt \qquad v=-\frac{1}{s}e^{-st}$$

Thus. $\vdots$

$$\mathcal{L}\{t^n\}(s)=\frac{n!}{s^{n+1}} \quad \text{for } n=1,2,\dots \text{ and } s>0$$