Homework 6 (Quiz 6)
Section 4.6
Section 4.7

Quiz on Wednesday...
study questions are already on the online homework website... I'll put the corresponding problem here from the book later today...

Definition:

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt =$$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty = \lim_{t\to\infty} \frac{1}{a-s} e^{(a-s)t} - \frac{1}{a-s} \boxed{e^{(a-s)\cdot 0}}^{\nearrow 1}$$

when does the limit exist? Tells us the domain of $\mathcal{L}\{e^{at}\}(s)$

If $s > a$ then $\lim_{t\to\infty} \frac{1}{a-s} e^{(a-s)t} = 0$ and

$$\mathcal{L}\{e^{at}\}(s) = -\frac{1}{a-s} = \frac{1}{s-a}$$

If $s \le a$ not in the domain because the limit doesn't converge...

(a) $\mathcal{L}\{1\} = \dfrac{1}{s}$ ☑

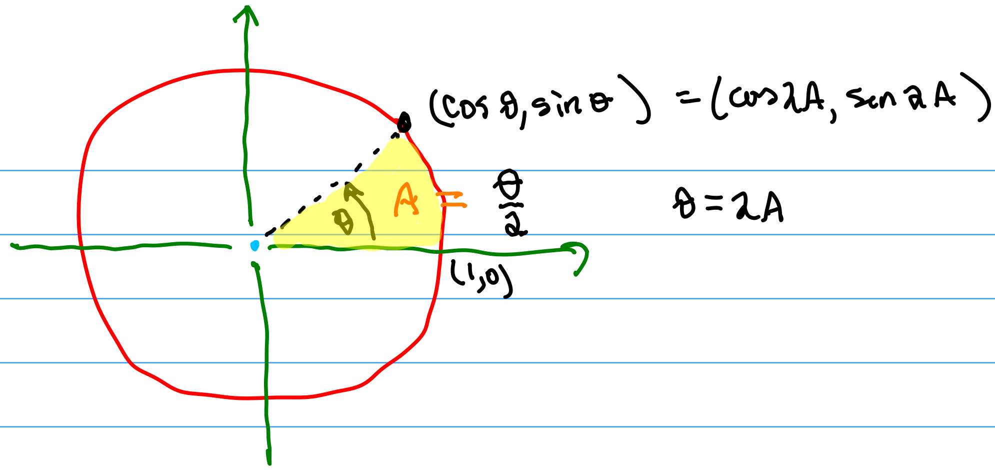
☑ (b) $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$ (c) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ ☑

? (d) $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$ ? (e) $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

$e^{i\theta} = \cos\theta + i\sin\theta$

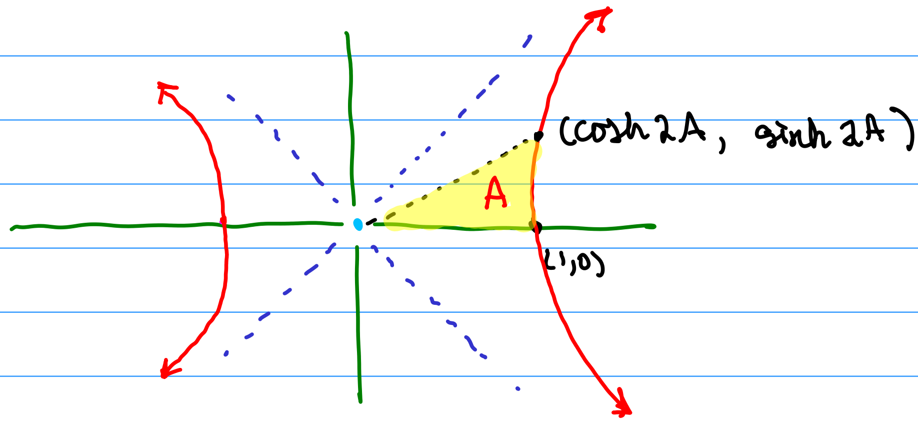
? (f) $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$ ? (g) $\mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

$x^2+y^2=1$




$\theta = 2A$

$x^2-y^2=1$




$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\mathcal{L}\{\sinh kt\}(s) = \int_0^\infty e^{-st}\sinh kt\,dt$$

$$= \int_0^\infty e^{-st}\,\frac{e^{kt}-e^{-kt}}{2}\,dt = \frac{1}{2}\int_0^\infty e^{-st}e^{kt}\,dt - \frac{1}{2}\int_0^\infty e^{-st}e^{-kt}\,dt$$

$$= \frac{1}{2}\mathcal{L}\{e^{kt}\}(s) - \frac{1}{2}\mathcal{L}\{e^{-kt}\}(s)$$

recall

$$\mathcal{L}\{e^{at}\}(s) = -\frac{1}{a-s} = \frac{1}{s-a}$$

$\mathcal{L}\{\sinh kt\}(s)$

$$= \frac{1}{2}\mathcal{L}\{e^{kt}\}(s) - \frac{1}{2}\mathcal{L}\{e^{-kt}\}(s)$$

$s > k$ $\quad$ $s > -k$ $\quad$ $s > \max(k, -k) = |k|$

$$= \frac{1}{2}\frac{1}{s-k} - \frac{1}{2}\frac{1}{s+k} = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right)$$

$$= \frac{1}{2}\frac{s+k-(s-k)}{(s-k)(s+k)} = \frac{k}{s^2-k^2}$$

domain is for $s > |k|$.

From the table...

$$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$$

We don't need to worry about the domains for s very much... because we'll translate our solutions back to t so it makes physical sense...

For regular sine and cosine

$e^{i\theta} = \cos\theta + i\sin\theta$ $\qquad$ $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

Thus

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$ $\quad$ or $\quad$ $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ $\quad$ or $\quad$ $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Similar to expressing $\sinh x$ and $\cosh x$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

domain is $s > 0$ ...

$$\mathcal{L}\{\cos kt\}(s) = \int_0^\infty e^{-st}\cos kt\,dt$$ exists for $s > 0$

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\,dt = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$u = e^{-st} \qquad dv = f'(t)\,dt$$

$$du = -se^{-st}\,dt \qquad v = f(t)$$

$$= e^{-st} f(t)\Big|_0^\infty + \int_0^\infty f(t)\, s\, e^{-st}\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f'(t)\}(s) = \underbrace{e^{-st} f(t)\Big|_0^\infty} + s\,\mathcal{L}\{f(t)\}(s)$$

simplify this .. usually only to 0 endpoint remains here . . .

This property with derivatives makes the Laplace transform useful for differential equations.