$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\,dt = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$u = e^{-st} \qquad dv = f'(t)dt$$

$$du = -se^{-st}dt \quad v = f(t)$$

$$= e^{-st} f(t)\Big|_0^\infty + \int_0^\infty f(t)\, s\, e^{-st}\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f'(t)\}(s) = \underbrace{e^{-st} f(t)\Big|_0^\infty}_{} + s\,\mathcal{L}\{f(t)\}(s)$$

$$= \underbrace{\lim_{t\to\infty} e^{-st} f(t)}_{} - e^{-s\cdot 0} f(0) + s\,\mathcal{L}\{f(t)\}(s)$$

assume this limit is zero for $s > 0$, this means f is not exponentially (or faster) increasing.

Under this assumption

$$\mathcal{L}\{f'(t)\}(s) = -f(0) + s\,\mathcal{L}\{f(t)\}(s)$$

already know

$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$

also know $f(t) = t$ then $f'(t) = 1$

plug in

$$\mathcal{L}\{1\}(s) = -0 + s\,\mathcal{L}\{t\}(s)$$

Solve for $\mathcal{L}\{t\}(s)$

$$\mathcal{L}\{t\}(s) = \frac{\mathcal{L}\{1\}(s)}{s} = \frac{1/s}{s} = \frac{1}{s^2}$$

Try again. set $f(t) = t^2$ $f'(t) = 2t$

$$\mathcal{L}\{f'(t)\}(s) = -f(0) + s\mathcal{L}\{f(t)\}(s)$$

$$\mathcal{L}\{2t\}(s) = -0^2 + s\mathcal{L}\{t^2\}(s)$$

$$2\mathcal{L}\{t\}(s) = s\mathcal{L}\{t^2\}(s)$$

$$\mathcal{L}\{t^2\}(s) = \frac{2\mathcal{L}\{t\}(s)}{s} = \frac{2\,1/s^2}{s} = \frac{2}{s^3}$$

Try again. set $f(t) = t^n$ $f'(t) = nt^{n-1}$

$$\mathcal{L}\{f'(t)\}(s) = -f(0) + s\mathcal{L}\{f(t)\}(s)$$

$$n\mathcal{L}\{t^{n-1}\}(s) = -0^n + s\mathcal{L}\{t^n\}(s)$$

Solve (after induction)

$$\mathcal{L}\{t^n\}(s) = \frac{n\mathcal{L}\{t^{n-1}\}(s)}{s} = \frac{n!}{s^{n+1}}$$

Use Laplace transform to transform a diff. eqn.

$$y'' + 2y' - 3y = 0$$

$$\mathcal{L}\{y'' + 2y' - 3y\}(s) = \mathcal{L}\{0\}(s) = 0$$

$$2(-y(0) + s\mathcal{L}\{y\}(s)) - 3\mathcal{L}\{y\}(s)$$

next time

$$\mathcal{L}\{f'(t)\}(s) = -f(0) + s\mathcal{L}\{f(t)\}(s)$$