$$y'' + 2y' - 3y = 0$$

$$\mathcal{L}\{y'' + 2y' - 3y\}(s) = \mathcal{L}\{0\}(s) = 0$$

next time

$$2\left(-y(0) + s\,\mathcal{L}\{y\}(s)\right) - 3\,\mathcal{L}\{y\}(s)$$

$$\mathcal{L}\{f'(t)\}(s) = -f(0) + s\,\mathcal{L}\{f(t)\}(s)$$

$$\mathcal{L}\{f''(t)\}(s) = -f'(0) + s\,\mathcal{L}\{f'(t)\}(s)$$

$$= -f'(0) + s\left(-f(0) + s\,\mathcal{L}\{f(t)\}(s)\right)$$

$$\mathcal{L}\{f''(t)\}(s) = -f'(0) - s f(0) + s^2\,\mathcal{L}\{f(t)\}(s)$$

$$\mathcal{L}\{y''(t)\}(s) = -y'(0) - s y(0) + s^2\,\mathcal{L}\{y(t)\}(s)$$

$y'' + 2y' - 3y = 0$ $\quad \boxed{y(0) = y_0 \quad y'(0) = y_1}$ conditions for unique solution...

$$\mathcal{L}\{y'' + 2y' - 3y\}(s) = \mathcal{L}\{0\}(s) = 0$$

$$-y'(0) - s y(0) + s^2\,\mathcal{L}\{y(t)\}(s) + 2\left(-y(0) + s\,\mathcal{L}\{y\}(s)\right) - 3\,\mathcal{L}\{y\}(s) = 0$$

$$(s^2 + 2s - 3)\,\mathcal{L}\{y\}(s) = (s+2)\,y(0) + y'(0)$$

$$\mathcal{L}\{y\}(s) = \frac{(s+2)\,y(0) + y'(0)}{s^2 + 2s - 3}$$

I know the Laplace transform of the solution...
What is y?

$$\frac{(s+2)y(0)+y'(0)}{s^2+2s-3} = \frac{s\,y_0}{s^2+2s-3} + \frac{\overbrace{2y_0+y_1}^{\text{constant}}}{s^2+2s-3}$$

Partial fractions decomposition...

$$\frac{2y_0+y_1+sy_0}{s^2+2s-3} = \frac{2y_0+y_1+sy_0}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

cross multiply...

$$2y_0+y_1+sy_0 = A(s-1) + B(s+3)$$

$$\begin{aligned} 2y_0+y_1 &= -A+3B \\ y_0 &= A+B \end{aligned}$$

$$3y_0+y_1 = 4B$$

so $B = \dfrac{3y_0+y_1}{4}$

$$\begin{aligned} 2y_0+y_1 &= -A+3B \\ -3y_0 &= -3A-3B \end{aligned}$$

$$-y_0+y_1 = -4A$$

$$A = \frac{y_0-y_1}{4}$$

$$\mathcal{L}\{y\}(s) = \frac{A}{s+3} + \frac{B}{s-1}$$

11. e^{at}	$\dfrac{1}{s-a}$

find y_1 so $\mathcal{L}\{y_1\}(s) = \dfrac{1}{s+3}$ and y_2 so $\mathcal{L}\{y_2\}(s) = \dfrac{1}{s-1}$

$$y_1 = e^{-3t} \qquad y_2 = e^{t}$$

Therefore the unique solution is...

$$y(t) = Ae^{-3t} + Be^{t} = \frac{y_0-y_1}{4}e^{-3t} + \frac{3y_0+y_1}{4}e^{t}$$

Laplace transforms of functions that can't be differentiated... Step function

$$\mathcal{U}(t) = \begin{cases} 1 & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases} \qquad \mathcal{U}(t-a) = \begin{cases} 1 & \text{for } t>a \\ 0 & \text{for } t<a \end{cases}$$

$$\mathcal{L}\{\mathcal{U}(t-a)\}(s) = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$ (assume $a>0$)

$$= \cancel{\int_0^a e^{-st}\,\mathcal{U}(t-a)\,dt} + \int_a^\infty e^{-st}\,\underbrace{\mathcal{U}(t-a)}_{1}\,dt$$

$$= \int_a^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_a^\infty$$

$$= \lim_{t\to\infty}\left(-\frac{1}{s}e^{-st}\right) + \frac{1}{s}e^{-sa} = \frac{e^{-sa}}{s}$$

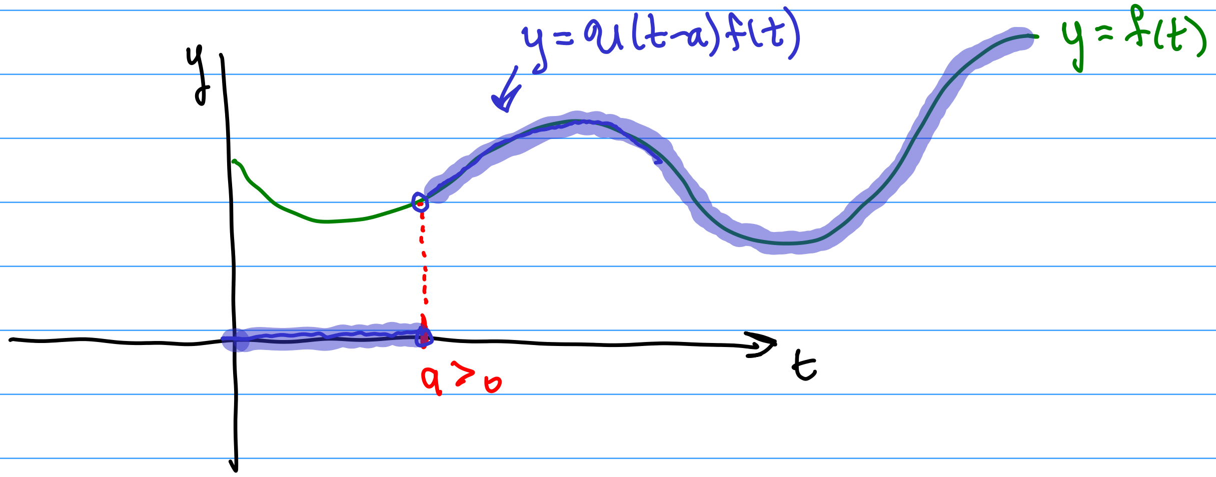
assume $s>0$

Therefore

$$\mathcal{L}\{\mathcal{U}(t-a)\}(s) = \frac{e^{-sa}}{s} = e^{-sa}\,\mathcal{L}\{1\}(s).$$

How about

$$\mathcal{L}\{\mathcal{U}(t-a)f(t)\}(s)$$



$$\mathcal{L}\{U(t-a)f(t)\}(s) = \int_0^\infty e^{-st}\, U(t-a) f(t)\, dt$$

$$= \int_a^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-s(u+a)} f(u+a)\, du$$

$u = t - a$

$du = dt$

$t = u + a$

$$= e^{-sa}\int_0^\infty e^{-su} f(u+a)\, du = e^{-sa}\int_0^\infty e^{-st} f(t+a)\, dt$$

$$= e^{-sa}\,\mathcal{L}\{f(t+a)\}(s)$$

$$\mathcal{L}\{U(t-a)f(t)\}(s) = e^{-sa}\,\mathcal{L}\{f(t+a)\}(s)$$

appears in table...

53.	$g(t)\mathscr{U}(t-a)$	$e^{-as}\mathscr{L}\{g(t+a)\}$

52.	$f(t-a)\mathscr{U}(t-a)$	$e^{-as}F(s)$

$e^{-as}\mathcal{L}\{f(t)\}(s)$

Put the shift first. If $g(t) = f(t-a)$ then $g(t+a) = f(t)$

THEOREM 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a).$$

$$\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

plays the role of s   set $r = s-a$

$$= \int_0^\infty e^{-\overbrace{(s-a)}t}f(t)\,dt = \int_0^\infty e^{-rt}f(t)\,dt$$

$$= \int_0^\infty e^{-rt}f(t)\,dt = \mathcal{L}\{f(t)\}(r) = \mathcal{L}\{f(t)\}(s-a)$$

Therefore, ...

$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a)$$