Laplace Transform of a periodic function

Suppose $f(t) = f(t+T)$ for some fixed $T > 0$.

Then

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

$$\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(u+T)} f(u+T)\,du = \int_0^\infty e^{-su} e^{-sT} f(u)\,du$$

$u = t - T \qquad t = u + T$

$dt = du$

e^{-sT} doesn't depend on u

$$= e^{-sT} \int_0^\infty e^{-su} f(u)\,du = e^{-sT} \mathcal{L}\{f(t)\}(s)$$

this is again Laplace transform of f

$$\mathcal{L}\{f(t)\}(s) = \int_0^T e^{-st} f(t)\,dt + e^{-sT} \mathcal{L}\{f(t)\}(s)$$

$$(1 - e^{-sT})\,\mathcal{L}\{f(t)\}(s) = \int_0^T e^{-st} f(t)\,dt$$

for comparison

$$\frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

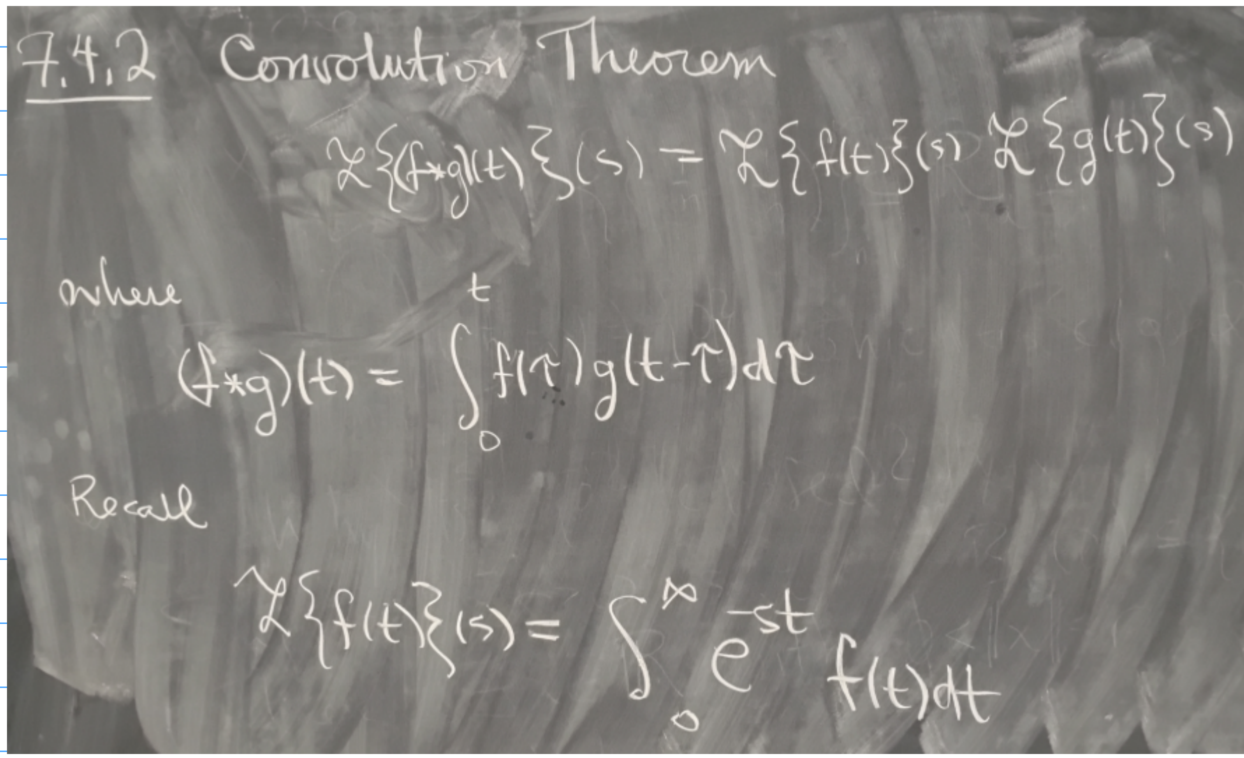
$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)\, e^{t-\tau}\, d\tau \quad ~~\text{for } f(t)~~$$

← use Laplace transform and convolution theorem to get f outside the integral...

Can use Laplace transform to solve this equation:

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{3t^2\}(s) - \mathcal{L}\{e^{-t}\}(s) - \mathcal{L}\left\{\int_0^t f(\tau)\, e^{t-\tau}\, d\tau\right\}(s)$$

this is a convolution...



$$\int_0^t f(\tau)\, e^{t-\tau}\, d\tau = (f * \exp)(t)$$

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{3t^2\}(s) - \mathcal{L}\{e^{-t}\}(s) - \mathcal{L}\{(f * \exp)(t)\}(s)$$

$$= \mathcal{L}\{3t^2\}(s) - \mathcal{L}\{e^{-t}\}(s) - \mathcal{L}\{f(t)\}(s)\, \mathcal{L}\{e^{t}\}(s)$$

$$\left(1+\mathcal{L}\{e^t\}(s)\right)\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{3t^2\}(s) - \mathcal{L}\{e^{-t}\}(s)$$

Therefore...

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{3t^2\}(s) - \mathcal{L}\{e^{-t}\}(s)}{1+\mathcal{L}\{e^t\}(s)}$$

Full solution...

$$f(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{3t^2\}(s) - \mathcal{L}\{e^{-t}\}(s)}{1+\mathcal{L}\{e^t\}(s)}\right\}(t)$$

Now simplify...

$n=2$ $a=-1$

$$\frac{\mathcal{L}\{3t^2\}(s) - \mathcal{L}\{e^{-t}\}(s)}{1+\mathcal{L}\{e^t\}(s)} = \frac{3\frac{2}{s^3} - \frac{1}{s+1}}{1+\frac{1}{s-1}} = \frac{\frac{6}{s^3} - \frac{1}{s+1}}{\frac{s-1}{s-1}+\frac{1}{s-1}} = \frac{\frac{6}{s^3} - \frac{1}{s+1}}{\frac{s}{s-1}}$$

$a=1$

$$= \frac{s-1}{s}\left(\frac{6}{s^3} - \frac{1}{s+1}\right) = \left(1-\frac{1}{s}\right)\left(\frac{6}{s^3} - \frac{1}{s+1}\right)$$

$$= \frac{6}{s^3} - \frac{1}{s+1} - \frac{6}{s^4} + \frac{1}{s(s+1)} = \frac{6}{s^3} - \frac{1}{s+1} - \frac{6}{s^4} + \frac{1}{s} - \frac{1}{s+1}$$

use partial fractions

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \qquad 1 = A(s+1) + Bs = (A+B)s + A$$

$$A=1 \qquad A+B=0 \qquad B=-1$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^3} - \frac{2}{s+1} - \frac{6}{s^4} + \frac{1}{s}\right\} =$$ look up in the table term by term...