7.5 THE DIRAC DELTA FUNCTION

$$\delta_a(t-t_0) = \begin{cases} 0 & \text{for} \quad t < t_0 - a \\ \frac{1}{2a} & \text{for} \quad t_0 - a \leq t < t_0 + a \\ 0 & \text{fo} \quad t \geq t_0 - a \end{cases}$$

Now what is the Laplace transform of δ_a?

$t_0 > a$ (take a limit as $a \to 0$ in a moment).

$$\mathcal{L}\{\delta_a(t-t_0)\}(s) = \int_0^\infty e^{-st}\,\delta_a(t-t_0)\,dt = \int_{t_0-a}^{t_0+a} e^{-st}\,\frac{1}{2a}\,dt$$

$$\frac{1}{-s}\,e^{-st}\,\frac{1}{2a}\Bigg|_{t_0-a}^{t_0+a} = \frac{-1}{2as}\left(e^{-s(t_0+a)} - e^{-s(t_0-a)}\right)$$

Thus

$$\mathcal{L}\{\delta_a(t-t_0)\}(s) = \frac{-1}{2as}\left(e^{-s(t_0+a)} - e^{-s(t_0-a)}\right)$$

Now consider the limit as $a \to 0$.

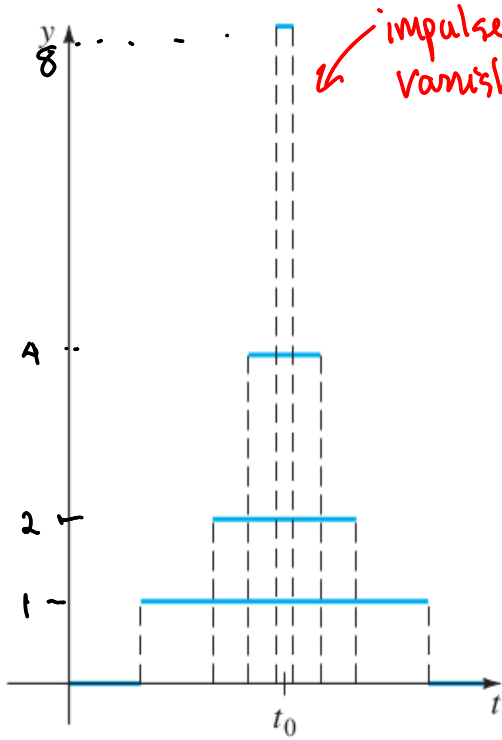
no matter what a is the area is 1.

$\frac{1}{2a}=8$ $a=\frac{1}{16}$

$\frac{1}{2a}=4$ $a=\frac{1}{8}$

$\frac{1}{2a}=2$ $a=\frac{1}{4}$

$\frac{1}{2a}=1$ $a=\frac{1}{2}$



(b) behavior of δ_a as $a \to 0$

impulse is application of a force over a vanishingly small amount of time...

Note as $a \to 0$ then.

$$\lim_{a\to 0} \delta_a(t-t_0) = \begin{cases} \infty & \text{for } t=t_0 \\ 0 & \text{for } t \neq t_0 \end{cases}$$

- can't put something that is infinite into a force of a differential equation.
- Use Laplace transform...

order is important so it makes sense

Transform and then take the limit

$$\lim_{a\to 0} \mathcal{L}\{\delta_a(t-t_0)\}(s) = \lim_{a\to 0} \frac{-1}{2as}\left(e^{-s(t_0+a)} - e^{-s(t_0-a)}\right)$$

$$= \frac{1}{2s}e^{-st_0} \lim_{a\to 0}\left(\frac{e^{sa}-e^{-sa}}{a}\right)$$

use l'hospital's rule to take limit since this is 0/0 indeterminate case.

$$\frac{\frac{d}{da}\left(e^{sa}-e^{-sa}\right)}{\frac{d}{da}(a)} = \frac{se^{sa}+se^{-sa}}{1} \to 2s \text{ as } a\to 0$$

Therefore

$$\lim_{a\to 0} \mathcal{L}\{\delta_a(t-t_0)\}(s) = \frac{1}{2s}e^{-st_0}\cdot 2s = e^{-st_0}$$

Rule 58 in the end of the book

58. $\delta(t-t_0)$ $\quad e^{-st_0}$

By definition of what the impulse $\delta(t-t_0)$ means...

$$\mathcal{L}\{\delta(t-t_0)\}(s) = \lim_{a\to 0} \mathcal{L}\{\delta_a(t-t_0)\}(s)$$

Note that $\mathcal{L}\{\lim_{a\to 0} \delta_a(t-t_0)\}(s)$ is different! so the limit of the transform and transform of the limit are different in this case...

57. $\delta(t)$ $\quad 1$

Also have

$$\mathcal{L}\{\delta(t)\}(s) = \mathcal{L}\{\lim_{t_0\to 0} \delta(t-t_0)\}(s) = \lim_{t_0\to 0} \mathcal{L}\{\delta(t-t_0)\}(s)$$

$$= \lim_{t_0\to 0} e^{-st_0} = e^{-s0} = 1$$

Not this

$$\mathcal{L}\{\delta_a(t)\}(s) = \int_0^\infty e^{-st}\delta_a(t)\,dt = \frac{1}{2a}\int_0^a e^{-st}\,dt$$

$$= \frac{-1}{2as} e^{-st}\Big|_0^a = \frac{-1}{2as}\left(e^{-sa}-1\right)$$

different!

$$\lim_{a\to 0} \frac{-1}{2as}\left(e^{-sa}-1\right) = \lim_{a\to 0}\frac{1}{2s}\left(\frac{1-e^{-sa}}{a}\right) = \lim_{a\to 0}\frac{1}{2s}\left(\frac{0+se^{-sa}}{1}\right)$$

$$= \frac{1}{2s}\lim_{a\to 0} se^{-sa} = \frac{1}{2}$$

7.5 Exercises

force that is an impulse... Next time

1. $y' - 3y = \delta(t - 2), \quad y(0) = 0$

2. $y' + y = \delta(t - 1), \quad y(0) = 2$

3. $y'' + y = \delta(t - 2\pi), \quad y(0) = 0, y'(0) = 1$

4. $y'' + 16y = \delta(t - 2\pi), \quad y(0) = 0, y'(0) = 0$

THEOREM 7.4.3 Transform of a Periodic Function

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

THEOREM 7.4.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

THEOREM 7.4.1 Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

THEOREM 7.3.2 Second Translation Theorem

If $F(s) = \mathscr{L}\{f(t)\}$ and a 0, then

$$\mathscr{L}\{f(t - a)\,\mathscr{U}(t - a)\} = e^{-as}F(s).$$

THEOREM 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at} f(t)\} = F(s - a).$$

THEOREM 7.2.2 Transform of a Derivative

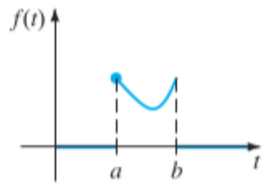
If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

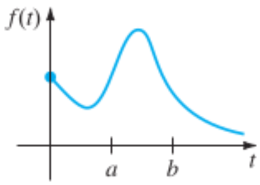
41. $\mathscr{L}\{\cos 2t\, \mathscr{U}(t - \pi)\}$ 42. $\mathscr{L}\left\{\sin t\, \mathscr{U}\left(t - \frac{\pi}{2}\right)\right\}$

51.



(f)

(f) $f(t - a)\,\mathscr{U}(t - a) - f(t - a)\,\mathscr{U}(t - b)$



$a = -\pi$

41. $\mathscr{L}\{\cos 2t\, \mathscr{U}(t - \pi)\}$ $= e^{-\pi s} \underbrace{\mathscr{L}\{\cos 2(t-\pi)\}(s)}$

now look this up in the other table...

$\mathscr{L}\{f(t - a)\,\mathscr{U}(t - a)\} = e^{-as}F(s).$

$f(t-a) = \cos 2t$

$f(t) = \cos 2(t+a)$

$\mathscr{L}\{\cos 2(t-\pi)\}(s) = \mathscr{L}\{\cos(2t - 2\pi)\}(s)$

cosine is 2π periodic

$= \mathscr{L}\{\cos 2t\}(s) = \dfrac{s}{s^2+4}$

from Theorem 7.1.1

Therefore

$\mathscr{L}\{\cos 2t\, \mathscr{U}(t-\pi)\}(s) = e^{-\pi s} \dfrac{s}{s^2+4}$