impulse force ...

7. $y'' + 2y' = \delta(t-1), \quad y(0) = 0, y'(0) = 1$

Solve by Laplace transforms. recall $\delta(t-1)$ is not a function but a function-like object (generalized function) that has been defined in terms of what its Laplace transform is.

$$\mathcal{L}\{\delta(t-1)\}(s) = e^{-s} \qquad \text{(by definition of } \delta)$$

Recall that the imaginary unit i was defined as that number-like object (imaginary number) such that

$$i^2 = -1$$

Another remark: $\delta(t-1)^2$ doesn't have a meaning.

$$\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{y'\}(s) = \underbrace{\mathcal{L}\{\delta(t-1)\}(s)}_{e^{-s}}$$

Theorem 7.2.2 $Y(s) = \mathcal{L}\{y\}(s)$

$n=2$ $\mathcal{L}\{y''\}(s) = s^2 Y(s) - s\,y(0) - y'(0)$

$n=1$ $\mathcal{L}\{y'\}(s) = sY(s) - y(0)$

recall $y(0) = 0, y'(0) = 1$

Therefore

$$s^2 Y(s) - s\,y(0) - \underbrace{y'(0)}_{1} + 2\left(sY(s) - y(0)\right) = e^{-s}$$

$$(s^2 + 2s)\,Y(s) = e^{-s} + 1$$

$$Y(s) = \frac{e^{-s}+1}{s^2+2s} = \frac{e^{-s}}{s(s+2)} + \frac{1}{s(s+2)}$$

second translation theorem 7.3.2.

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \underbrace{e^{-as}}_{e^{-s}} \; \underbrace{F(s)}_{\frac{1}{s(s+2)}}$$

what is this

$a=1$

$$F(s) = \mathcal{L}\{f(t)\}(s) = \frac{1}{s(s+2)}$$

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \qquad A(s+2) + Bs = 1$$

$$(A+B)s + 2A = 1 \qquad 2A = 1 \quad \text{and} \quad A+B=0$$

$$A = 1/2 \qquad B = -1/2$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1/2}{s}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1/2}{s+2}\right\}(t)$$

$a=-2$

Theorem 7.2.1

$$= \frac{1}{2}\cdot 1 - \frac{1}{2}e^{-2t} = \frac{1}{2}\left(1-e^{-2t}\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+2)}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\}(t)$$

$$= f(t-1)\mathcal{U}(t-1) + f(t)$$

$$= \frac{1}{2}\left(1-e^{-2(t-1)}\right)\mathcal{U}(t-1) + \frac{1}{2}\left(1-e^{-2t}\right)$$

since

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \underbrace{e^{-as}}_{e^{-s}} \; \underbrace{F(s)}_{\frac{1}{s(s+2)}}$$

$a=1$