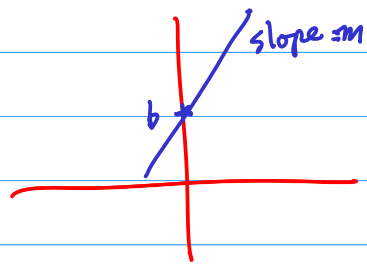
Matrices...

linear function $y = mx + b$



not a linear function by the definition here

not what we mean...

Any function that satisfies two algebraic properties

$$\boxed{\begin{cases} f(u) + f(v) = f(u+v) \\ f(\alpha v) = \alpha f(v) \quad \text{for any } \alpha \in \mathbb{R}, \end{cases}}$$

$$(mu + b) + (mv + b) = m(u+v) + 2b$$

$$m(u+v) + b = m(u+v) + b$$

different unless $b = 0$

Consider the function $f(x) = mx$. Then

$$f(u) + f(v) = (mu) + (mv) = m(u+v) = f(u+v)$$

$$f(\alpha v) = m\alpha v = \alpha(mv) = \alpha f(v)$$

Complicated examples when working with multiple variables.

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x + 5y.$$

$$f\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = 2u_1 + 5u_2 \qquad f\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = 2v_1 + 5u_2$$

$$f\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + f\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = (2u_1 + 5u_2) + (2v_1 + 5u_2)$$

associative and commutative properties of addition, distributive property of mult.

$$= 2(u_1 + v_1) + 5(u_2 + v_2) = f\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

essentially why vector addition is defined like it is.

Represent $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ using a table

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x + 5y = \underbrace{\begin{bmatrix} 2 & 5 \end{bmatrix}}_{\text{represents the function } f.} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix}}_{\text{represents the function } g.} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y + 6z \\ 4x + 5y + 7z \end{bmatrix}$$

$$(f \circ g)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = f\left(g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right) = f\left(\begin{bmatrix} x + 3y + 6z \\ 4x + 5y + 7z \end{bmatrix}\right)$$

$$= 2(x + 3y + 6z) + 5(4x + 5y + 7z)$$

$$= (2 + 5 \cdot 4)x + (2 \cdot 3 + 5 \cdot 5)y + (2 \cdot 6 + 5 \cdot 7)z$$

$$= \underbrace{\begin{bmatrix} (2 + 5 \cdot 4) & (2 \cdot 3 + 5 \cdot 5) & (2 \cdot 6 + 5 \cdot 7) \end{bmatrix}}_{\text{represents the function } f \circ g.} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix} = \boxed{\begin{bmatrix} (2\cdot 1 + 5\cdot 4) & (2\cdot 3 + 5\cdot 5) & (2\cdot 6 + 5\cdot 7) \end{bmatrix}}$$

Find the pattern...

$$\begin{matrix} & \begin{bmatrix} 1 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix} \\ \begin{bmatrix} 2 & 5 \end{bmatrix} & \boxed{\begin{bmatrix} 2\cdot 1+5\cdot 4 & 2\cdot 3+5\cdot 5 & 2\cdot 6+5\cdot 7 \end{bmatrix}} \end{matrix}$$

$$\begin{array}{r} 12 \\ 35 \\ \hline 47 \end{array}$$

$$\begin{matrix} & \begin{bmatrix} 1 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix} \\ \begin{bmatrix} 2 & 5 \end{bmatrix} & \boxed{\begin{bmatrix} 22 & 31 & 47 \end{bmatrix}} \end{matrix}$$

Multiplication of matrices is the composition of the corresponding linear functions...

DEFINITION II.6 Multiplication of Matrices

Let **A** be a matrix having m rows and n columns and **B** be a matrix having n rows and p columns. We define the **product AB** to be the $m \times p$ matrix

$$\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & \cdots & a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2n}b_{np} \\ \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1p} + a_{m2}b_{2p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

$$= \left(\sum_{k=1}^{n} a_{ik}b_{kj} \right)_{m\times p}.$$

Systems of differential equations

$$\begin{cases} \frac{dy_1}{dt} = g_1(t, y_1, y_2) \\ \frac{dy_2}{dt} = g_2(t, y_1, y_2) \end{cases}$$

coupled equations ...

Write as vectors

$$X = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad G(t, X) = G\left(t, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} g_1(t, y_1, y_2) \\ g_2(t, y_1, y_2) \end{bmatrix}$$

$$\frac{d\underset{\sim}{X}}{dt} = \begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} g_1(t, y_1, y_2) \\ g_2(t, y_1, y_2) \end{bmatrix} = G(t, X)$$

Thus

$$\frac{d\underset{\sim}{X}}{dt} = G(t, X)$$

represents a system of differential equation as a vector equation

Focus on linear differential equations in Chapter 8.

$$G(t, X) = \underbrace{\overset{\text{matrix}}{A}\ \overset{\text{vector}}{X}}_{\text{linear function}} + F(t)$$

non linear function does not depend on the X were solving for

$$\frac{dX}{dt} = A(t)X + F(t) \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

square matrix

$$A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & a_{1n} \\ a_{21}(t) & a_{22}(t) & a_{23} & \cdots & a_{2n} \\ a_{31}(t) & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & \cdots & \cdots & \cdot & a_{nn} \end{bmatrix}$$

$$\boxed{\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + \cancel{f_1(t)}\\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + \cancel{f_2(t)}\\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + \cancel{f_n(t)}.\end{aligned}}$$

Homogeneous problem $F(t) = 0$.

$$\boxed{\frac{dX}{dt} = A(t)X}$$

Case where coefficients don't depend on time

$$\frac{dX}{dt} = AX$$

vector version of the first problem we solved.

$$\frac{dy}{dx} = 5y$$

separation of variables

$$\int \frac{dy}{y} = \int 5\,dx$$

$\ln|y| = 5x + c_1$

$|y| = e^{5x+c_1} = e^{c_1}e^{5x}$

$y = \pm e^{c_1} e^{5x}$

$\boxed{y = e^{5x} c_2}$ $\quad c_2 = \pm e^{c_1}$

$$\frac{dX}{dt} = AX$$

I'd like to say $X = e^{At} c$

give meaning to this

on Monday we talk more about this first and what that means...