Case where coefficients don't depend on time

$$\frac{dX}{dt} = AX$$

vector version of the first problem we solved.

A is a square $n \times n$ matrix of real number

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$ where $a_{ij} \in \mathbb{R}$

set of real numbers

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \\ x_n(t) \end{bmatrix} \quad \text{so} \quad \frac{dX}{dt} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \\ x_n'(t) \end{bmatrix}$$

First part of Linear algebra is spent solving linear equations:

$$AX = b$$

no derivative...
b is just a vector of numbers.

Idea, use method of elimination to solve for X.

Second part solve the eigenvalue/eigenvector problem (non-linear)

$$AK = \lambda K$$ where K is a vector and λ is a scalar

n-dim

just a real number

solve for K and λ and they appear as a quadratic term on the right side...

Many ways to find λ and K. For example, the theory of determinants was used in math 330 (if you took that). Iterative approximations that converge in the limit are used numerically for large matrices.

Solving this problem writes a matrix-vector multiplication as a scalar-vector multiplication.

Matrix-vector multiplication

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} \\ \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} \end{bmatrix}$$

$$AK = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix} = \begin{bmatrix} a_{11}k_1 + a_{12}k_2 + \cdots + a_{1n}k_n \\ a_{21}b_1 + a_{22}k_2 + \cdots + a_{2n}k_n \\ \vdots \\ a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nn}k_n \end{bmatrix}$$

$$\lambda K = \lambda \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix} = \begin{bmatrix} \lambda K_1 \\ \lambda K_2 \\ \vdots \\ \lambda K_n \end{bmatrix}$$ much simpler than this

Back to the linear differential equation.

$$\frac{dX}{dt} = AX$$ and suppose $AK = \lambda K$

plug in

$$X(t) = Ke^{\lambda t}$$ λ is a scalar so $e^{\lambda t}$ is easy to understand.

$$\frac{dX}{dt} = \frac{d}{dt} Ke^{\lambda t} = K \frac{d}{dt} e^{\lambda t} = K \lambda e^{\lambda t} = \lambda K e^{\lambda t} = \lambda X$$

scalar

$$AX = A Ke^{\lambda t} = (AK)\, e^{\lambda t} = \lambda K e^{\lambda t} = \lambda X$$

Therefore $\frac{dX}{dt} = AX$ when $X(t) = Ke^{\lambda t}$ so that's a solution!

Example: $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ Find eigenvalues and eigenvectors ...

Check these are eigenvectors

$$K_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$AK_1 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = (-1)K_1 \quad \text{so} \quad \lambda_1 = -1$$

use a diagram

$$\begin{array}{cc} & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} & \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \end{array}$$

$$AK_2 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (-1)K_2 \quad \text{so} \quad \lambda_2 = -1$$

$$AK_3 = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix} = (5)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 5K_3 \quad \text{so} \quad \lambda_3 = 5$$

$$\begin{array}{cc} & \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix} \end{array}$$

We have three solutions one for each eigenvalue eigenvector pair

$$X_1 = K_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t} \qquad X_2 = K_2 e^{\lambda_2 t} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-t} \qquad X_3 = K_3 e^{\lambda_3 t} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5t}$$

General solution is

Since all X_i are solutions to $\frac{dX}{dt} = AX$

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 X_8(t)$$

$$X'(t) = c_1 X_1'(t) + c_2 X_2'(t) + c_3 X_3'(t)$$

$$= c_1 A X_1(t) + c_2 A X_2(t) + c_3 A X_3(t)$$

since there are just real numbers

$$= A c_1 X_1(t) + A c_2 A_2(t) + A c_3 X_3(t)$$

$$= A\left(c_1 X_1(t) + c_2 A_2(t) + c_3 X_3(t)\right) = AX(t)$$

Thus $\frac{dX}{dt} = AX$.

used both properties of linear functions

$$\begin{cases} f(u) + f(v) = f(u+v) \\ f(\alpha v) = \alpha f(v) \quad \text{for any } \alpha \in \mathbb{R}. \end{cases}$$

for factoring the A out

for the c_i's moving inside.

So now what to do with an initial condition

$$\frac{dX}{dt} = AX \quad \text{such that} \quad X(0) = X_0 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

General solution

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5t}$$

$$X(0) = c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

solve for c_1, c_2, c_3 interms of α, β, γ.

Equivalently.

$$\begin{aligned} c_1 + c_3 &= \alpha \\ c_1 + c_2 - c_3 &= \beta \\ c_2 + c_3 &= \gamma \end{aligned}$$

eliminate, substitute and so forth...

Back to the vector formulation

X_0

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$K = \left[\, K_1 \,\vdots\, K_2 \,\vdots\, K_3 \,\right] \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad X_0 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Then $\boxed{\underbrace{Kc} = X_0}$ and this is linear algebra problem to solve for c.

this is a linear function of c

$$g(c) = Kc \qquad \boxed{g(c) = X_0} \xrightarrow{\text{solve for } c} c = g^{-1}(X_0)$$

Notation is that $g^{-1}(X_0) = K^{-1} X_0$

is the matrix for the inverse function g^{-1}.