Matrices represent linear functions

Consider

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 0y + 0z \\ 0x + 1y + 0z \\ 0x + 0y + 1z \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{I} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Matrix-vector multiplication means evaluate the corresponding linear function.

Finding eigenvectors and eigenvalues.

Square matrix $A \in \mathbb{R}^{n \times n}$

\# columns

\# rows

each entry is a real number

linear function for matrix A.

$f(X) = AX$

eigenvector is a vector K such that $AK = \lambda K$ for some scalar λ.

$f(K) = \lambda K$ means this

Idea is given a matrix A find K and λ. Thus, solve

$$AK = \lambda K \quad \text{for } K \text{ and } \lambda.$$

quadratic.

① Iterative approximation of the solution on a computer

works for very large matrices..

② Theory of determinants.

useful for 2x2 and 3x3 matrices

$$AK = \lambda K$$

$$AK = \lambda I K$$

$$AK - \lambda I K = 0$$

$$\underbrace{(A - \lambda I)}_{\text{matrix } B} K = 0$$

want a K that is non-zero...

Theory of determinants says the only time $BK = 0$ has a solution $K \neq 0$ is when $\det B = 0$.

Find values of λ such that

$$\det(A - \lambda I) = 0$$

What are the det of a 2x2 and 3x3 matrix?

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{ad - bc}_{2 \text{ terms } = 2!}$$

4 entries

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \underbrace{aei + bfg + cdh - ceg - bdi - afh}_{6 \text{ terms } = 3!}$$

9 entries

$$\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & j \\ m & n & o & p \end{bmatrix} = 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \text{ terms}$$

16 entries

Only use determinants for 2x2 and 3x3 matrices... larger ones use a approx. method with a computer...

Example:

$$\boxed{\begin{aligned}\frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= x + 3y\end{aligned}}$$

linear function

write in vector form

$$\frac{dX}{dt} = AX$$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$ so that $\frac{dX}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$

$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ so $AX = \begin{bmatrix} 2x + 2y \\ x + 3y \end{bmatrix}$

Now solve for λ

$$\det(A - \lambda I) = 0$$

$$\det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) - 2\cdot 1 = \lambda^2 - 5\lambda + 6 - 2$$

$$= \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

so $\lambda = 1$ and $\lambda = 4$ are the eigenvalues

To find eigenvectors, plug λ's back into $AK = \lambda K$ and solve for the K's.