$\frac{dX}{dt} = AX$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

Now solve for λ

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) - 2\cdot 1 = \lambda^2 - 5\lambda + 6 - 2$$

$$= \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

so $\lambda = 1$ and $\lambda = 4$ are the eigenvalues

To find eigenvectors, plug λ's back into $AK = \lambda K$ and solve for the K's.

$(A - \lambda I)K = 0$ and solve for $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$\lambda_1 = 1$ // $A - 1\cdot I = \begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

$k_1 + 2k_2 = 0$

$k_1 + 2k_2 = 0$

If the two equations are not linearly independent, this indicates a mistake in finding λ.

Since 2 unknowns and only one equation there are an infinite # of solutions.

$k_1 + 2k_2 = 0$ set $k_2 = 1$ and solve for $k_1 = -2k_1 = -2$

solution is $K_2 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

- Idea find a solution K that doesn't involve fractions for easier hand calculation.
- Alternatively: find a K that is a unit vector so $|K| = \sqrt{k_1^2+k_2^2} = 1$

$\lambda_2 = 4$

$$A - 4I = \begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \qquad \begin{aligned} -2k_1 + 2k_2 &= 0 \\ 1k_1 - 1k_2 &= 0 \end{aligned}$$

check the eqns are lin. dep and choose one to solve.

$1k_1 - 1k_2 = 0$ If $k_2 = 1$ then $k_1 = k_2 = 1$ so $K_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

In summary,

$$\lambda_1 = 1, \; K_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 4, \; K_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are the eigenvalue/eigenvector pairs for the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

Trying to solve $\frac{dX}{dt} = AX$. Now I have two solutions

$$X_1(t) = K_1 e^{\lambda_1 t} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} \quad \text{and} \quad X_2(t) = K_2 e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}.$$

General solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}.$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

HW 8.2 #20

$$\frac{dX}{dt} = AX \quad \text{where} \quad A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$$

Find eigenvalues and eigenvectors

two minus signs

$$\det(A-\lambda I) = \det\begin{bmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{bmatrix} = (-6-\lambda)(4-\lambda) - (5)(-5)$$

$$= (\lambda-4)(\lambda+6)+25 = \lambda^2+2\lambda-24+25 = \lambda^2+2\lambda+1$$

$$= (\lambda+1)^2 = 0 \quad \text{so} \quad \lambda = -1$$

Plug λ back into $(A-\lambda I)K=0$ and solve for K.

$\lambda = -1$

$$A + I = \begin{bmatrix} -6+1 & 5 \\ -5 & 4+1 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \qquad \underbrace{-5k_1+5k_2=0}_{\text{simplify}} \quad k_1 = k_2$$

setting $k_2 = 1$ gives solution $K_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

first independent solution

$$X_1(t) = \boxed{K_1 e^{\lambda_1 t}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

How to find another solution?

$$X_2(t) = tK_1 e^{\lambda_1 t} + Pe^{\lambda_1 t}$$

product with the previous solution

solve for P, but we'll already assume the form of the term needed!

Plug $X_2(t)$ in and solve for P so it's a solution.

$$\frac{dX_2}{dt} = \frac{d}{dt}\left(tK_1 e^{\lambda_1 t} + Pe^{\lambda_1 t}\right)$$

$$= K_1 e^{\lambda_1 t} + \boxed{t\lambda_1 K_1 e^{\lambda_1 t}} + \lambda_1 P e^{\lambda_1 t} =$$

$$= A\left[\underset{\text{eigenvector of } A}{tK_1} e^{\lambda_1 t} + Pe^{\lambda_1 t}\right] = \boxed{t\lambda_1 K e^{\lambda_1 t}} + APe^{\lambda_1 t}$$

Thus

$$K_1 \cancel{e^{\lambda_1 t}} + \lambda_1 P \cancel{e^{\lambda_1 t}} = AP \cancel{e^{\lambda_1 t}}$$

$$K_1 + \lambda_1 P = AP \qquad \text{solve for } P$$

$$AP - \lambda_2 P = K_1$$

$$\underbrace{(A - \lambda_1 I)}_{\text{not invertible...}} P = K_1$$

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-5p_1 + 5p_2 = 1$$

$$-5p_1 + 5p_2 = 1$$

$$-p_1 + p_2 = \frac{1}{5}$$

$$p_1 = p_2 - \frac{1}{5}$$

Take $p_2 = 1$ then $p_1 = 1 - \frac{1}{5} = \frac{4}{5}$ so $P = \begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$

$$X_2(t) = tK_1 e^{\lambda_1 t} + Pe^{\lambda_1 t}$$

$$= t\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 4/5 \\ 1 \end{bmatrix} e^{-t}$$

General solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 4/5 \\ 1 \end{bmatrix} e^{-t} \right)$$

Example: (pg 242)

$$\frac{dX}{dt} = AX \quad \text{where } A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

Eigenvalues and eigenvectors:

$$\det(A - \lambda I) = \det \begin{bmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix} = (6-\lambda)(4-\lambda) + 5 = 0$$

get complex values of λ.

$$\lambda^2 - 10\lambda + 24 + 5 = \lambda^2 - 10\lambda + 29 = 0$$

$a = 1 \quad b = -10 \quad c = 29$

$$\begin{array}{r} {}^{3}29 \\ \underline{4} \\ 116 \end{array}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 - 116}}{2} = \frac{10 \pm i\sqrt{16}}{2} = 5 \pm 4i$$